Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

Minimizing Attention in a Motion Control Context

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***Abstract*— In this paper we motivate the need for an algorithmic solution to the problem of segmenting and representing vector-valued functions of time with the goal of identifying a set of standard parts which can be used as the basis for a efficient description of an entire family. We motivate the use of such an approach in a feedback control context where one can use such efficient descriptions together with concatenation and scaling operations to generate quite general trajectories. Our results include a procedure for jointly specifying feedback gains and open-loop controls leading to the definition of efficient motion control languages.**

I. Introduction

In [1] we gave a mathematical definition of attention suitable for use in a control context and discussed its use as a means to establish a trade-off between the quality of the trajectories and the cost of implementing the control law. In this paper we build on these ideas, now turning more specifically to problems arising in robotics. More specifically, there is a growing literature devoted to the theory and experimental results concerned with motion control languages. These languages go beyond the basic pick and place robotic languages developed in the 1970's in that they incorporate the capability of scheduling feedback gains as an integral part of the language. Reference [2] is an early paper in this direction. More recent results appear in the work done at Maryland [3] and Berkeley [4]. In their most primitive forms, these languages steer systems of the form

$$\dot{x} = f(x) + g(x)u$$

using strings of triples of the form (u, k, T) which cause the system to follow trajectories that satisfy

$$\dot{x}(t) = f(x(t)) + g(x(t))k(x(t)) + g(x(t))u(t) \qquad ; t_i \leq t \leq t_i + T.$$

The motivating force behind this development is based on the observation that the use of such languages allows one to rein in the complexity associated with specifying trajectories in a high dimensional space while achieving a degree of robustness with respect to the effects of environmental change. However, part of the preliminary work required to make good use of this approach is the construction of a suitable "dictionary" of such triples. In a biological setting it might be argued that suitable families of these (u, k, T)'s would be learned from trial and error. In an engineering setting it is possible to reason in a similar way. We envision an off line algorithm which analyzes successful paths, originally generated from a variational principle or a human operator, with a view toward finding simpler description through smoothing and merging segments. The specification of such procedures is the subject of this paper.

Of course segmentation is an important part of the analysis of signals arising in other areas such as speech and video. In speech it has been studied using hidden Markov models; in computer vision, where there is a somewhat less universal and/or clear performance measure, a variety of techniques have been used. In all cases the goals are similar; simplify through segmentation and tokenization. The techniques investigated here can be thought of being in the spirit of Grenander's treatise, *General Pattern Theory* [5]. Important steps in learning the suitable primitives for describing a language involve merging and parsing of an observed sequence into segments. We propose one such parsing algorithm here. Our basic approach could be described as a modified principal components algorithm in which we allow a warping of the time axis along with the usual projection onto maximal energy subspaces.

II. Minimal Attention and the Liouville Equation

Associated with the control model

$$\dot{x} = f(x) + ug(x)$$

is a second evolution equation, the linear partial differential equation

$$\frac{\partial \rho(t,x)}{\partial t} = \langle \nabla, (f + ug)\rho(t,x) \rangle.$$

This is usually called the Liouville equation. If we let $\rho(0,x)$ be a distribution of initial conditions on x, then the value of ρ at time t gives the corresponding distribution at time t. For example, if x has an initial distribution of values defined by $\rho_0(\cdot)$ and if x satisfies the linear equation

$$\dot{x} = Ax + bu$$

then the solution of the Liouville equation is

$$\rho(t,x) = \frac{1}{e^{\mathrm{tr}At}} \rho_0 \left(e^{-At}x - \int_0^t e^{A(t-\sigma)} bu(\sigma) d\sigma \right).$$

*This work was supported in part by the National Science Foundation under Yale prime CCR 9980058 and the US Army Research Office under Boston University prime GC169369 NGD, the National Science Foundation grant number EIA 0218411.

One interpretation is to think of ρ as being a probability distribution evolving in time under the deterministic flow defined by $\dot{x} = f(x) + g(x)u$. This is consistent with the observation that the Liouville equation is just the Fokker-Planck equation for the Itô equation

$$dx = (f(x) + g(x)u)dt + b\, dw$$

with b set equal to zero. However, in principle, there is nothing stochastic about the Liouville equation.

For our purposes it is significant that the Liouville equation, like the Fokker-Planck equation, is sensitive to the distinction between expressing u in open loop form verses expressing it in some x-dependent way, even though the solution of the ordinary differential equation $\dot{x} = f(x) + g(x)u$ for a single initial condition is indifferent to such a change.

With this background and notation we can consider the problem of choosing a control law for $\dot{x} = f(x) + ug(x)$ which is easy to implement in the sense that it makes only modest requirements on the sampling rate and is insensitive to quantization errors in the data paths. As in [1], we propose to capture these ideas using an attention functional of the form

$$\eta_a = \int_{[0,\infty)} \int_{\mathbb{R}^n} L_a \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \right) dx\, dt.$$

Specific choices of interest include the family

$$\eta_a = \int_{[0,\infty)} \int_{\mathbb{R}^n} a \left(\frac{\partial u}{\partial t} \right)^2 + b \left(\frac{\partial u}{\partial x} \right)^2 dx\, dt.$$

In order to assure that the trajectories will be well behaved, we include in the performance measure a more conventional trajectory dependent term of the form

$$\eta_p = \int_{[0,\infty]} \int_{\mathbb{R}^n} L_p(x,u)\, \rho(t,x)\, dx\, dt.$$

Because we want the feedback control law to work for a range of initial conditions, we postulate a density of initial conditions $\rho_0(x)$ which evolves according to the Liouville equation given above.

As contrasted with a standard optimal control problem formulation we can make the following points:

1) The cost of control law implementation and the distinction between open and closed loop implementations is captured by this formulation.
2) The addition of a penalty on the sizes of the partial derivatives of u will make the solutions less sensitive to changes caused by approximate implementations involving sampling and quantization.
3) On the negative side, we must now deal with an evolution equation that is a partial differential equation, even though the individual trajectories for the system under consideration are described by an ordinary differential equation.

III. Open loop verses Closed Loop

As discussed in the introduction, an important element of the motion description languages under discussion here is the fact that the motion is being described using feedback. The purpose of the feedback term is to simplify the description of the control and to give the system the ability to adjust to the environment. It is essential in most situations to constrain the set of possible feedback laws because of limitations on the gain or a desire for stability. The constraints on the open loop part of the control can be more directly determined by the desire for a simple description. This suggests the following class of problems.

The Optimal Segment Specification Problem: Given a system $\dot{x} = f(x,u)\ ;\ y = h(x)$ and given a class of admissible feedback control laws, $\mathcal{K}$, find $k \in \mathcal{K}$ and v such that the solution of $\dot{x} = f(x, k(x) + v)$ generates a pre specified desired trajectory on $[T_{i-1}, T_i]$ and $||\dot{v}||$ is minimized.

The following lemma describes a setting in which the splitting of the control into a feedback term and an open loop term can solved algorithmically.

Lemma: Suppose we are given a m-input, m-output controllable, observable and invertible linear system

$$\dot{x} = Ax + Bu\ \ ;\ \ y = Cx$$

and an admissible class of feedback controls of the form $k(x) = K(x - x_0)$ with K coming from a convex set $\hat{K}$. Then given a desired output $y_d(\cdot)$ defined on $[T_i, T_{i+1}]$ which can be achieved through choice of a differentiable input u, the choice of (K, x_0, v) that generates the desired trajectory while minimizing

$$\gamma = \int_{T_i}^{T_{i+1}} \left\| \frac{d}{dt}(u - Kx - Kx_0) \right\|^2 dt$$

can be found by solving a finite dimensional convex programming problem.

Proof: Notice that because the linear system is invertible, there is a unique input to $\dot{x} = Ax + bu$ that achieves the trajectory and there is a unique state trajectory as well. Thus if we express the control as $u = K(x - x_0) + v$ we can write γ as

$$\gamma = \int_{T_i}^{T_{i+1}} ||\dot{u} - K\dot{x}||^2 dt$$

Because both u and x are known, we can integrate to get a functional of the form

$$\gamma = a + \mathrm{tr}KB + \mathrm{tr}(KQK)$$

with K constrained to a convex set. ∎

IV. Modified Principal Component Analysis

Having an algorithm to minimize the attention requirements over a single segment of the motion, we can now turn to the problem of finding clusters of language elements with a view toward simplifying the dictionary when this will not lead to a loss of expressiveness. To this end we investigate a suitable modification of principle component analysis.

In the usual approach to principal component analysis one begins with a fixed inner product space and a collection of vectors in that space. Assuming that the vectors have sample mean zero, one forms the sample covariance matrix

$$W = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T$$

and finds the eigenvectors and eigenvalues of W. Because the eigenvalues of a symmetric, nonnegative definite matrix are all real and nonnegative, we can arrange the eigenvalues in order, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$. We can expand each of the x's in terms of the corresponding normalized eigenvectors, which we denote by $f_1, f_2, ..., f_n$,

$$x_i = \sum_{i=1}^{n} a_{ij} f_j$$

If the eigenvalues are all about the same size, ignoring the contribution of any one of the eigenvectors will result in a significant error. On the other hand, if one or more of the of the eigenvalues are much smaller than the others, then we can drop their contribution to the expansions without introducing a large error. More precisely,

$$\sum_{i=1}^{n} ||x_i - \sum_{i=1}^{n'} a_{ij}\eta_j||^2 \leq \sum_{i=n'+1}^{n} \lambda_i$$

Recalling that the eigenvalues are assumed to be arranged in decreasing order, we introduce the quantity

$$m = \frac{1}{\sum \lambda_i}\sum_{i=1}^{n} i\lambda_i$$

to measure the rapidity with which the sequence $\lambda_1 \geq \lambda_2 ...$ falls off. Clearly $1 \leq m \leq n(n+1)/2$, with the lower limit corresponding to the case W has only one nonzero eigenvalue and the upper limit corresponding to the case where all the eigenvalues are equal. We refer to m as the *first moment* of the eigenvalue sequence. Clearly it measures some kind of average performance associated with the entire range of trade-offs one might make between accuracy and complexity. Obviously the main interest in principal component analysis comes from the situations in which m/n is small.

Principle component analysis often fails even on data with apparent common features because of an inability to align the samples. This is particularly true when it is applied in a function space setting. In contemplating its use in clustering (u, k, T) elements, this is particularly true because there is no aprori choice of the time interval. The following refinement of PCA is sometimes of interest. Let $\{x_i\}$ be as above and let $\mathcal{G}$ be a Lie group. Let $\phi : \mathcal{G} \times \mathbb{R}^n \to \mathbb{R}^n$ be a group action. By the PCA$_\phi$ problem, we understand the problem of finding a sequence of G's, $G_1, G_2, ...G_m$ such that the first moment associated with the vectors $\phi(G_1, x_1), \phi(G_2, x_2), ...\phi(G_n, x_n)$ is as small as possible. Some aspects of this are similar to the work in [6].

The PCA$_\phi$ Algorithm: Given the collection $\{x_i\}$, define

$$W_{\{G_i\}} = \sum_{i=1}^{n} \phi(G_i, x_i)\phi^T(G_i, x_i)$$

Starting with all the G_i set equal to the identity element of $\mathcal{G}$, choose G_1 such that $m(W_{\{G_i\}})$ is minimized. Repeat with G_2 and continue, cycling through the entire set with successive adjustments to minimize $m(W_{\{G_i\}})$. Clearly m remains the same or decreases at each step. It is bounded from below and so repeated cycling leads to a stationary point which is a local minimum.

Notice that this procedure works directly with the sample variance and does not require the identification of the coefficients of expansion. It also does not require an aprori choice as to how many principle components will be used in the final expansion. This can be decided after the optimal G-scaling has been done.

The set of differentiable monotone increasing functions mapping the interval $[T_{i-1}, T_i]$ into itself form a group under composition. Thus the following problem fits into the class described above.

The Optimal Warping Problem: Suppose that we have n functions, $\{x_k(\cdot)\}$ defined on $[0, T]$. Find a collection of n monotone increasing functions $\{\phi_k\}$ that map $[0, T]$ into itself and have the property that the mean of the sample variance matrix is minimized.

Remark: It would be desirable to have an algorithm for solving versions of the PCA$_\phi$ problem in which the cost function included terms that penalized the group elements in proportion to their distance from the identity. In the optimal warping problem one might incorporate the penalty term

$$\eta = \int (\phi^{(1)}(t) - 1)^2 dt \leq \epsilon$$

V. The Synthesis

We have argued that the use of complex systems in complex environments can be made more practical by the use of suitable motion description languages. At the highest level, the system is conceptualized as a transducer that accepts symbolic descriptions of the tasks to be done and then complies these into detailed control laws. The compilation process involves identifying a suitable segmentation of the trajectories and the identification of a splitting of the individual control laws into an open loop and closed loop pieces. Because we want to keep overall operation as simple as possible, it is important to incorporate a complexity penalty at each stage of the optimization. The use of attention minimization is designed to achieve this goal. The elements of the motion description language are to be found empirically. Based on human generated, or off line computed, desirable trajectories, we propose the use of a scalable version of principle component analysis to identify suitable building blocks of the language. Experimental work incorporating aspects of this program will be reported in future papers.

VI. REFERENCES

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