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Control Theory and Singular Riemannian Geometry*

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This paper discusses the qualitative and quantitative aspects of the solution of a class of optimal control problems, together with related questions concerning a corresponding stochastic differential equation. The class has been chosen to reveal what one may expect for the structure of the set of conjugate points for smooth problems in which existence of optimal trajectories is not an issue but for which Lie bracketing is necessary to reveal the reachable set. It is, perhaps, not too surprising that in thinking about this problem various geometrical analogies are useful and, in the final analysis, provide a convenient language to express the results. Indeed, the geodesic problem of Riemannian geometry is commonly taken to be the paradigm in the calculus of variations; a point of view which is supported by a variety of variational principles such as the theorem of Euler which identifies the path of a freely moving particle on a manifold with a geodesic and the whole theory of general relativity. Nonetheless, the class of variational problems considered here can only be thought of as geodesic problems in some limiting sense in which the metric tends to infinity. For this reason the geodesic analogy has to be developed rather carefully. What is actually needed is a generalization of Riemannian geometry and it seems that the intuitive content of Riemannian geometry is sufficiently robust so as to withstand modifications of the type required and still provide a reasonably "geometric" picture. We consider questions involving model spaces, geodesic equations, the appropriate definition of the Laplace–Beltrami operator, etc. The end results make avail-

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able in the control setting, considerable geometrical insight and suggest some novel problems in differential geometry.

In addition to work in control theory and geometry which we draw on in a very specific way, one sees in the recent work of physicists an exciting, albeit vague, parallelism centering around the idea of "superspace." Before embarking on the actual mathematics of this paper let me make a few comments on this. Part and parcel of the Riemannian expression for infinitesimal distance

$$(ds)^2 = \Sigma g_{ij}(x) dx_i dx_j$$

is idea that space is "essentially isotropic." That is to say, the distance to nearby points involves the same kind of expression regardless of the direction. Characteristic of the models which we investigate here is a very strong anisotropic character as would be suggested by an expression such as $(ds)^2 = (dx)^2 + (dy)^2 + |dz|$. There have been, and continue to be, suggestions in the physics literature to the effect that what we perceive as being a four dimensional space-time continuum may be better thought of as being a submanifold of a higher dimensional space. In the theory of O. Klein and T. Kaluza (see [1]) one takes the ambient space to be five dimensional, obtaining in return a setting in which electromagnetic and gravitational theories are unified. In more recent work, e.g., Zumino's article in [2], one sees suggestions about ten and twenty six dimensional ambient spaces. Manifestly these theories refer to a highly anisotropic kind of space. Having planted the idea that what is to be discussed here may have physical as well as mathematical interest we hasten to add that only the mathematical and control theoretic aspects will be considered further.

Optimal control and geodesics have been discussed before in the literature, for example Hermes [3] and Hermann [4], however the most directly relevant prior work that I am aware of occurs in the thesis of J. Baillieul [5] where he carries out certain detailed computations on a specific model of the type considered here.

I thank the organizers of the conference for giving me the opportunity to speak at my alma mater on the occasion of its 100th anniversary. It was a pleasant occasion. I also want to express my appreciation to J. Baillieul, C. I. Byrnes and N. Gunther for their patience in listening to, and help in clarifying the arguments given here.

The Starting Point

Consider a neighborhood of x_0 in n -dimensional Cartesian space \mathbb{R}^n , and consider the following problem from control theory. Given

$$\dot{x} = B(x)u, \quad \dot{x} = \frac{d}{dt}x$$

find $u(t) \in \mathbb{R}^m$ on the interval $[0, 1]$ such that $x(0) = x_0$, $x(1) = x_1$, and

$$\eta(x_0, x_1) = \int_0^1 (\langle u, u \rangle)^{1/2} dt$$

is minimized. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^m . We investigate this problem under the assumption that B is smooth and of constant rank m . In place of η we study

$$\rho(x, y) = \min_u \eta(x, y).$$

Notice that ρ satisfies the condition $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x) > 0$ if $x \neq y$ and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. That is, ρ satisfies the axioms of a metric. The only step here which is not completely obvious is $\rho(x, y) = \rho(y, x)$ and this is proven by replacing $u_i(t)$ by $-u_i(1-t)$ and noticing that this control steers y to x if u steers x to y .

In the special case where $m = n$, under our announced hypotheses we may rewrite $\dot{x} = B(x)u$ as $B^{-1}(x)\dot{x} = u$ and express the problem as a Riemannian geodesic problem, i.e., to find from among all smooth paths joining x and y the one which minimizes

$$\eta = \int_0^1 (\langle B^{-1}(x)\dot{x}, B^{-1}(x)\dot{x} \rangle)^{1/2} dt.$$

Thus we see that $(B^{-1}(x))^T B^{-1}(x) = G(x)$ plays the role of the metric tensor if $B^{-1}(x)$ exists. However, $\rho(x, y)$ may be well defined even if B is not invertible and in particular even if $m < n$. All that is needed for $\rho(x, y)$ to be defined is that every point should be reachable from every other point. None of the phenomena which we investigate are a consequence of any lack of smoothness in B or the quantity being minimized; for the sake of simplicity we take B to be C^∞ although we could get by with less.

What are the conditions for every point near x to be reachable from x ? This kind of question is studied in the control literature under the names controllability, reachability, etc. but the specific result we need was known already by Chow [6] who generalized a result of Caratheodory. What is needed is that the Lie algebra of vector fields generated by

$$F_i = \sum_{j=1}^m b_j^i \frac{\partial}{\partial x^j}, \quad B = (b_j^i)$$

should be sufficiently rich to span \mathbb{R}^n at each point. This condition is considerably less demanding than the condition that B is invertible!

Perhaps an example will be of some help in developing intuition. Consider the following prototype for the situation in \mathbb{R}^3 :

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= uy - vx. \end{aligned}$$

In this case

$$F_u = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}; \quad F_v = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad [F_u, F_v] = 2 \frac{\partial}{\partial z}.$$

Since these span \mathbb{R}^3 we can reach any point from any other point. However, B is 3 by 2 and so $B^T B$ is not invertible and we are not in the standard Riemannian situation. With the help of the Lagrange multiplier technique one can show that the geodesics satisfy

$$\begin{aligned} \ddot{x} + \lambda \dot{y} &= 0 \\ \ddot{y} - \lambda \dot{x} &= 0 \\ \ddot{z} + \lambda(\dot{x}x + \dot{y}y) &= 0 \end{aligned}$$

where λ is a suitable constant. In fact, from the last equation we see that for trajectories which pass through $(0, 0, z)$ we have

$$\lambda = \frac{\dot{z}}{x^2 + y^2}.$$

The locus of points equidistant from $(0, 0, 0)$ displays an x_3 -axis symmetry but, in contrast with the Riemannian situation, the geodesic spheres are not smooth manifolds. (They fail to be smooth at the north and south poles.)

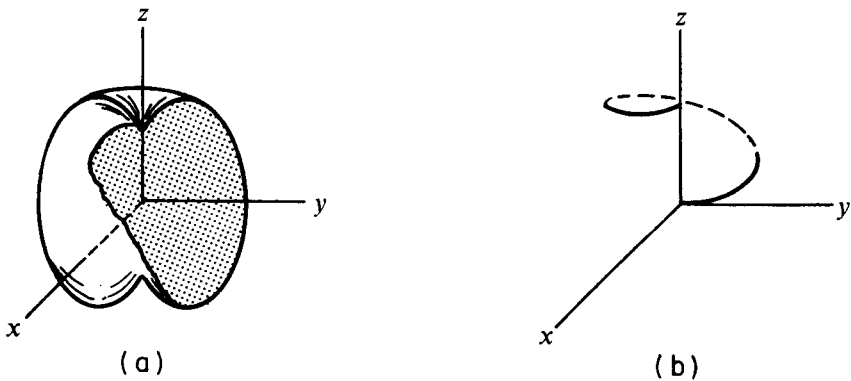


Figure 1 The geodesic spheres and one geodesic curve.

We can think of this example in the following way. At each point in the space we have a two dimensional subspace of the tangent space, the one spanned by the vector fields

$$\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}.$$

In this plane we have a given inner product corresponding to the fact that we are minimizing the integral of $u^2 + v^2$. We may think of this plane as being a

two space of *ordinary directions*. In this problem the geodesics emanating from a point are characterized by an initial velocity chosen from the ordinary directions together with parameter λ which controls, in a way we want to make precise, the amount of twist the trajectory has to bring it away from the plane of ordinary directions.

It may also be pointed out that for this example the points conjugate to the point $(0, 0, 0)$ consist of the entire z -axis. Recall [7] that in an ordinary Riemannian space the points conjugate to p have distance

$$\rho(p, q) \geq \frac{\pi}{\sqrt{K}}$$

where K is the maximum sectional curvature of the manifold. Since p is conjugate to points in every neighborhood of it we see that we are dealing with a space having rather exceptional curvature!

Naturally associated with this problem is a subgroup of the affine group on \mathbb{R}^3 consisting of elements of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \theta_{11} & \theta_{12} & 0 \\ \theta_{21} & \theta_{22} & 0 \\ \alpha & \beta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -\beta \\ +\alpha \\ \gamma \end{bmatrix}, \quad \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \text{orthogonal}.$$

This group acts transitively on \mathbb{R}^3 and leaves the form of the variational problem invariant. Thus the calculation of $\rho(\cdot, \cdot)$ is no more difficult than the calculation of $\rho(0, \cdot)$. Based on this remark we can see that just as through $(0, 0, 0)$ there is a line of points $\{p | p = (0, 0, z)\}$ which are conjugate to $(0, 0, 0)$, there is a line of points $\{p | p = (a, b, z)\}$ which are conjugate to (a, b, z_0) . At each point in \mathbb{R}^3 this gives us a natural splitting of the tangent space into a two dimensional subspace Range B and a one dimensional subspace defined by tangent vector to the manifold of conjugate points.

Finally, there is a second order operator associated with this problem, namely

$$L = \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right)^2 + \left(\frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \right)^2,$$

which shares many of the properties of the heat operator. We will discuss this further in the final section of the paper.

The Hamiltonian Formulation

We now return to the general situation and set about the problem of studying the geodesics. It saves a certain amount of annoying calculation to

observe right at the start that the trajectories which minimize η also minimize

$$\tilde{\eta} = \int_0^1 \langle u, u \rangle dt.$$

This comes about, as it does in the case of Riemannian geometry, because the value of $\langle u, u \rangle$ along geodesic curves is constant.

In Riemannian geometry the equations for the geodesics can be written as equations on the tangent bundle. Choosing coordinates, these may be expressed in terms of the Levi-Civita connection as

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0 \quad (\text{summation convention}).$$

In the present situation the tangent bundle formulation is not quite so straightforward. Instead, we begin with a Hamiltonian formulation on the cotangent bundle. According to the maximum principle of optimal control Hamilton-Jacobi theory in the present context we may associate with the geodesic problem a pair of first order equations

$$\dot{x} = Bu$$

$$\dot{p} = A(u, p),$$

where A is a bilinear form in u and p , and assert that if $x(\cdot)$ is a geodesic then there exists a $p(0)$ such that (x, p) satisfy these equations with

$$A(u, p) = -\frac{\partial}{\partial x} p^T B u$$

and

$$u = B^T p.$$

Geometrically, the pair (x, p) is to be thought of as a point in the cotangent bundle T^*X . In this setup each geodesic through x is generated by a choice of $p(0) \in T_x^*X$ but, just as in Riemannian geometry where one does not know *a priori* which values of $\dot{x}(0)$ generate paths over $[0, 1]$ without cut points, here we are not sure *a priori* which values of $p(0)$ generate curves which are free of cut points on $[0, 1]$.

In order to prevent one from attempting to attach intrinsic meaning to an accidental choice of coordinates it is worthwhile to recast these ideas in coordinate free and, while we are at it, global terms. Let X be a manifold and let \tilde{E} be a rank m euclidean vector bundle over X . Let $B: \tilde{E} \rightarrow E \subset TX$ be a vector bundle isomorphism. If $\langle \cdot, \cdot \rangle$ is the inner product on \tilde{E} then the sub-bundle of TX defined by E has an inner product which comes from $\langle \cdot, \cdot \rangle$. Associated with E is a sequence of derived distributions. Define E_x^0 as $\text{span } B(x)$ and continue inductively

$$E^{(1)} = \overset{\text{span}}{=} (E_x^{(0)} + [E_x^{(0)}, E_x^{(0)}]), \quad E^{(2)} = \overset{\text{span}}{=} (E^{(1)} + [E^{(1)}, E^{(1)}]), \dots, \text{etc.},$$

where the brackets indicate vector fields which arise as Lie brackets of vector fields in the space indicated. If the dimensions of $E_x^{(i)}$ are, for each i , independent of x then E defines a sequence of derived vector bundles $E^{(0)} \subset E^{(1)} \subset E^{(2)} \subset \dots$. The condition for the system to be controllable is that this sequence should terminate at TX . Of course E determines, canonically, a dimension m subbundle $E^\dagger \subset T^*X$, $E^\dagger = \{p \mid p \text{ vanishes on } E\}$. The map $B: \tilde{E} \rightarrow E$ and the inner product define a map from T^*X/E^\dagger into \tilde{E} which is given in coordinates by $p \mapsto B^T p = u$. The pair of equations given above then define a section of the tangent bundle of T^*X . If the controllability condition is satisfied then we get a metric $\rho(\cdot, \cdot)$ on X and we may be sure that any two points in X are joined by a geodesic.

We also point out the following additional result which plays a role later. Suppose that $E^{(1)}$ equals TX . In that case the inner product structure on E can be used to define an inner product on $([E, E] + E)/E$. The idea is analogous to the one whereby an inner product on the space of one forms is used to define an inner product the space of two forms, etc. This goes as follows. Let b_1, b_2, \dots, b_m be an orthonormal basis for E in some neighborhood $U \subset X$. Any point in $([E, E] + E)/E$ can be then expressed as

$$X = \sum \alpha_{ij}[b_i, b_j] + E.$$

Such a representation is not unique, but among all such representations there is a unique one which minimizes

$$\left(\sum_{ij=1}^m \alpha_{ij}^2 \right)^{1/2} = \eta(X).$$

This then gives a mapping from $([E, E] + E)/E$ into $\mathbb{R}^{m(m-1)/2}$. It is easily seen to be linear. We define the length of a point in $([E, E] + E)/E$ as the minimum value of $\eta(X)$. It is easy to verify that this defines a norm and that the norm satisfies the parallelogram identity and so it comes from an inner product. Finally, one can check that the norm is independent of the choice of orthonormal basis.

Geodesic Equations

In order to better understand the qualitative behavior of the solutions of the optimal control problem which we introduced in the second section, we now describe a transformation which may be thought of as a partial inverse Legendre transformation. The effect of this transformation is to introduce as many second order equations as possible. Everything here is local.

Given the control equations $\dot{x} = B(x)u$, we then have a subbundle $E = \text{span } B$ in TX . In a neighborhood of any point x_0 we can find an

integrable subbundle \hat{E} of TX which is tangent to E at x_0 . In local coordinates this amounts to saying that we can write the given equations as

$$\dot{x}_u = B_u u$$

$$\dot{x}_l = B_l u$$

with B_u being an m by m nonsingular matrix and $B_l(x_0) = 0$. For each choice of integrable subbundle \hat{E} tangent to E at x_0 we get such a decomposition of the equations of motion by letting x_u be such that

$$\frac{\partial}{\partial x_u^1}, \dots, \frac{\partial}{\partial x_u^m}$$

span the integrable subbundle. As noted, E also determines a subbundle $E^\dagger \subset T^*X$, namely the subbundle of one forms which vanish on E . Denoting a typical point in E^\dagger by p_l we can write the equations of the previous section as

$$\dot{x}_u = B_u u$$

$$\dot{x}_l = B_l u$$

$$\dot{p}_u = A_{uu}(u, p_u) + A_{ul}(u, p_l)$$

$$\dot{p}_l = A_{lu}(u, p_u) + A_{ll}(u, p_l).$$

Differentiating the equation $\dot{x}_u = B_u B_u^T p_u + B_u B_l^T p_l$ with respect to time and using the differential equations for p we get a second order equation in x_u . By using $\dot{x}_u = B_u u$ to eliminate u we then end up with a pair of equations of the form

$$\begin{aligned} \ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + \Lambda_{jk}^i \dot{x}^j p^k &= 0 & x^i \in \{x^1, x^2, \dots, x^n\} \\ \dot{p}^i + F_{jk}^i \dot{x}^j \dot{x}^k + E_{jk}^i \dot{x}^j p^k &= 0 & p^i \in \{p^{m+1}, p^{m+2}, \dots, p^n\} \end{aligned}$$

where the coefficients depend on x but not \dot{x} or p . These equations have to be integrated along with the nonholonomic constraints represented by $\dot{x}_l = B_l B_u^{-1} \dot{x}_u$. The symmetries are as follows: Γ_{jk}^i is symmetric in jk and Λ_{jk}^i is skew symmetric in ij .

Since we did not give a canonical way to choose \hat{E} we cannot attach an intrinsic meaning to any aspect of these equations which is not invariant with respect to that choice. However, given any such choice, B_u defines an inner product on E and hence B_u defines a Riemannian structure on the submanifold passing through x_0 and defined by \hat{E} . When we change \hat{E} or x_0 we change this Riemannian structure. We call the original system *reducible* if there exists a choice for \hat{E} such that when we write $\dot{x}_u = B_u u$, B_u is of the form

$$B_u(x_u, x_l) = B_0(x_u) \theta(x_u, x_l)$$

with θ orthogonal. Under this circumstance the Riemannian structure does