

Reprint from

## New Directions in Applied Mathematics

Edited by Peter J. Hilton and Gail S. Young

---

© 1981 Springer-Verlag New York Inc.  
Printed in the United States of America. Not for Sale



Springer-Verlag  
New York Heidelberg Berlin



# Control Theory and Singular Riemannian Geometry\*

R. W. Brockett†

This paper discusses the qualitative and quantitative aspects of the solution of a class of optimal control problems, together with related questions concerning a corresponding stochastic differential equation. The class has been chosen to reveal what one may expect for the structure of the set of conjugate points for smooth problems in which existence of optimal trajectories is not an issue but for which Lie bracketing is necessary to reveal the reachable set. It is, perhaps, not too surprising that in thinking about this problem various geometrical analogies are useful and, in the final analysis, provide a convenient language to express the results. Indeed, the geodesic problem of Riemannian geometry is commonly taken to be the paradigm in the calculus of variations; a point of view which is supported by a variety of variational principles such as the theorem of Euler which identifies the path of a freely moving particle on a manifold with a geodesic and the whole theory of general relativity. Nonetheless, the class of variational problems considered here can only be thought of as geodesic problems in some limiting sense in which the metric tends to infinity. For this reason the geodesic analogy has to be developed rather carefully. What is actually needed is a generalization of Riemannian geometry and it seems that the intuitive content of Riemannian geometry is sufficiently robust so as to withstand modifications of the type required and still provide a reasonably "geometric" picture. We consider questions involving model spaces, geodesic equations, the appropriate definition of the Laplace-Beltrami operator, etc. The end results make avail-

\* This research was supported in part by the Army Research Office under Grant DAAG29-76-C-0139, the U.S. Office of Naval Research under the Joint Services Electronics Program Contract N00014-75-C-0648 and the National Science Foundation under Grant ENG-79-09459 at the Division of Applied Sciences, Harvard University, Cambridge, MA.

† Division of Applied Sciences, Harvard University, Cambridge, MA 02138.