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Control Theory and Singular Riemannian Geometry*

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This paper discusses the qualitative and quantitative aspects of the solution of a class of optimal control problems, together with related questions concerning a corresponding stochastic differential equation. The class has been chosen to reveal what one may expect for the structure of the set of conjugate points for smooth problems in which existence of optimal trajectories is not an issue but for which Lie bracketing is necessary to reveal the reachable set. It is, perhaps, not too surprising that in thinking about this problem various geometrical analogies are useful and, in the final analysis, provide a convenient language to express the results. Indeed, the geodesic problem of Riemannian geometry is commonly taken to be the paradigm in the calculus of variations; a point of view which is supported by a variety of variational principles such as the theorem of Euler which identifies the path of a freely moving particle on a manifold with a geodesic and the whole theory of general relativity. Nonetheless, the class of variational problems considered here can only be thought of as geodesic problems in some limiting sense in which the metric tends to infinity. For this reason the geodesic analogy has to be developed rather carefully. What is actually needed is a generalization of Riemannian geometry and it seems that the intuitive content of Riemannian geometry is sufficiently robust so as to withstand modifications of the type required and still provide a reasonably "geometric" picture. We consider questions involving model spaces, geodesic equations, the appropriate definition of the Laplace–Beltrami operator, etc. The end results make avail-

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able in the control setting, considerable geometrical insight and suggest some novel problems in differential geometry.

In addition to work in control theory and geometry which we draw on in a very specific way, one sees in the recent work of physicists an exciting, albeit vague, parallelism centering around the idea of "superspace." Before embarking on the actual mathematics of this paper let me make a few comments on this. Part and parcel of the Riemannian expression for infinitesimal distance

$$(ds)^2 = \Sigma g_{ij}(x)\, dx_i\, dx_j$$

is idea that space is "essentially isotropic." That is to say, the distance to nearby points involves the same kind of expression regardless of the direction. Characteristic of the models which we investigate here is a very strong anisotropic character as would be suggested by an expression such as $(ds)^2 = (dx)^2 + (dy)^2 + |dz|$. There have been, and continue to be, suggestions in the physics literature to the effect that what we perceive as being a four dimensional space–time continuum may be better thought of as being a submanifold of a higher dimensional space. In the theory of O. Klein and T. Kaluza (see [1]) one takes the ambient space to be five dimensional, obtaining in return a setting in which electromagnetic and gravitational theories are unified. In more recent work, e.g., Zumino's article in [2], one sees suggestions about ten and twenty six dimensional ambient spaces. Manifestly these theories refer to a highly anisotropic kind of space. Having planted the idea that what is to be discussed here may have physical as well as mathematical interest we hasten to add that only the mathematical and control theoretic aspects will be considered further.

Optimal control and geodesics have been discussed before in the literature, for example Hermes [3] and Hermann [4], however the most directly relevant prior work that I am aware of occurs in the thesis of J. Baillieul [5] where he carries out certain detailed computations on a specific model of the type considered here.

I thank the organizers of the conference for giving me the opportunity to speak at my alma mater on the occasion of its 100th anniversary. It was a pleasant occasion. I also want to express my appreciation to J. Baillieul, C. I. Byrnes and N. Gunther for their patience in listening to, and help in clarifying the arguments given here.

The Starting Point

Consider a neighborhood of x_0 in n-dimensional Cartesian space $\mathbb{R}^n$, and consider the following problem from control theory. Given

$$\dot{x} = B(x)u, \qquad \dot{x} = \frac{d}{dt}x$$

find $u(t) \in \mathbb{R}^m$ on the interval $[0, 1]$ such that $x(0) = x_0$, $x(1) = x_1$, and

$$\eta(x_0, x_1) = \int_0^1 (\langle u, u \rangle)^{1/2} \, dt$$

is minimized. Here $\langle \ , \ \rangle$ denotes the standard inner product on $\mathbb{R}^m$. We investigate this problem under the assumption that B is smooth and of constant rank m. In place of η we study

$$\rho(x, y) = \min_u \eta(x, y).$$

Notice that ρ satisfies the condition $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x) > 0$ if $x \neq y$ and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. That is, ρ satisfies the axioms of a metric. The only step here which is not completely obvious is $\rho(x, y) = \rho(y, x)$ and this is proven by replacing $u_i(t)$ by $-u_i(1 - t)$ and noticing that this control steers y to x if u steers x to y.

In the special case where $m = n$, under our announced hypotheses we may rewrite $\dot{x} = B(x)u$ as $B^{-1}(x)\dot{x} = u$ and express the problem as a Riemannian geodesic problem, i.e., to find from among all smooth paths joining x and y the one which minimizes

$$\eta = \int_0^1 (\langle B^{-1}(x)\dot{x}, B^{-1}(x)\dot{x} \rangle)^{1/2} \, dt.$$

Thus we see that $(B^{-1}(x))^T B^{-1}(x) = G(x)$ plays the role of the metric tensor if $B^{-1}(x)$ exists. However, $\rho(x, y)$ may be well defined even if B is not invertible and in particular even if $m < n$. All that is needed for $\rho(x, y)$ to be defined is that every point should be reachable from every other point. None of the phenomena which we investigate are a consequence of any lack of smoothness in B or the quantity being minimized; for the sake of simplicity we take B to be C^∞ although we could get by with less.

What are the conditions for every point near x to be reachable from x? This kind of question is studied in the control literature under the names controllability, reachability, etc. but the specific result we need was known already by Chow [6] who generalized a result of Caratheodory. What is needed is that the Lie algebra of vector fields generated by

$$F_i = \sum_{i=1} b_j^i \frac{\partial}{\partial x^j}, \qquad B = (b_j^i)$$

should be sufficiently rich to span $\mathbb{R}^n$ at each point. This condition is considerably less demanding than the condition that B is invertible!

Perhaps an example will be of some help in developing intuition. Consider the following prototype for the situation in $\mathbb{R}^3$:

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = uy - vx.$$

In this case

$$F_u = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}; \qquad F_v = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \qquad [F_u, F_v] = 2\frac{\partial}{\partial z}.$$

Since these span $\mathbb{R}^3$ we can reach any point from any other point. However, B is 3 by 2 and so B^TB is not invertible and we are not in the standard Riemannian situation. With the help of the Lagrange multiplier technique one can show that the geodesics satisfy

$$\ddot{x} + \lambda\dot{y} = 0$$

$$\ddot{y} - \lambda\dot{x} = 0$$

$$\ddot{z} + \lambda(\dot{x}x + \dot{y}y) = 0$$

where λ is a suitable constant. In fact, from the last equation we see that for trajectories which pass through $(0, 0, z)$ we have

$$\lambda = \frac{\dot{z}}{x^2 + y^2}.$$

The locus of points equidistant from $(0, 0, 0)$ displays an x_3-axis symmetry but, in contrast with the Riemannian situation, the geodesic spheres are not smooth manifolds. (They fail to be smooth at the north and south poles.)

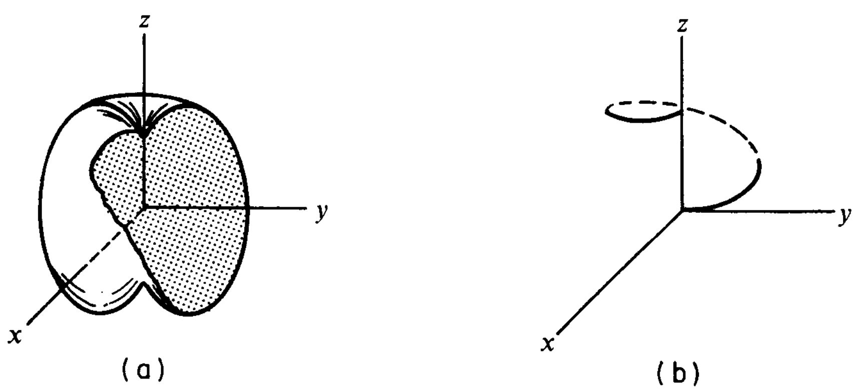

Figure 1 The geodesic spheres and one geodesic curve.

We can think of this example in the following way. At each point in the space we have a two dimensional subspace of the tangent space, the one spanned by the vector fields

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

In this plane we have a given inner product corresponding to the fact that we are minimizing the integral of $u^2 + v^2$. We may think of this plane as being a two space of *ordinary directions*. In this problem the geodesics emanating from a point are characterized by an initial velocity chosen from the ordinary directions together with parameter λ which controls, in a way we want to make precise, the amount of twist the trajectory has to bring it away from the plane of ordinary directions.

It may also be pointed out that for this example the points conjugate to the point (0, 0, 0) consist of the entire z-axis. Recall [7] that in an ordinary Riemannian space the points conjugate to p have distance

$$\rho(p, q) \geq \frac{\pi}{\sqrt{K}}$$

where K is the maximum sectional curvature of the manifold. Since p is conjugate to points in every neighborhood of it we see that we are dealing with a space having rather exceptional curvature!

Naturally associated with this problem is a subgroup of the affine group on $\mathbb{R}^3$ consisting of elements of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \theta_{11} & \theta_{12} & 0 \\ \theta_{21} & \theta_{22} & 0 \\ \alpha & \beta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -\beta \\ +\alpha \\ \gamma \end{bmatrix}, \qquad \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \text{orthogonal}.$$

This group acts transitively on $\mathbb{R}^3$ and leaves the form of the variational problem invariant. Thus the calculation of $\rho(\cdot, \cdot)$ is no more difficult than the calculation of $\rho(0, \cdot)$. Based on this remark we can see that just as through (0, 0, 0) there is a line of points $\{p \mid p = (0, 0, z)\}$ which are conjugate to (0, 0, 0), there is a line of points $\{p \mid p = (a, b, z)\}$ which are conjugate to (a, b, z_0). At each point in $\mathbb{R}^3$ this gives us a natural splitting of the tangent space into a two dimensional subspace Range B and a one dimensional subspace defined by tangent vector to the manifold of conjugate points.

Finally, there is a second order operator associated with this problem, namely

$$L = \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)^2,$$

which shares many of the properties of the heat operator. We will discuss this further in the final section of the paper.

The Hamiltonian Formulation

We now return to the general situation and set about the problem of studying the geodesics. It saves a certain amount of annoying calculation to observe right at the start that the trajectories which minimize η also minimize

$$\tilde{\eta} = \int_0^1 \langle u, u \rangle \, dt.$$

This comes about, as it does in the case of Riemannian geometry, because the value of $\langle u, u \rangle$ along geodesic curves is constant.

In Riemannian geometry the equations for the geodesics can be written as equations on the tangent bundle. Choosing coordinates, these may be expressed in terms of the Levi–Civita connection as

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \quad \text{(summation convention)}.$$

In the present situation the tangent bundle formulation is not quite so straightforward. Instead, we begin with a Hamiltonian formulation on the cotangent bundle. According to the maximum principle of optimal control Hamilton–Jacobi theory in the present context we may associate with the geodesic problem a pair of first order equations

$$\dot{x} = Bu$$

$$\dot{p} = A(u, p),$$

where A is a bilinear form in u and p, and assert that if $x(\cdot)$ is a geodesic then there exists a $p(0)$ such that (x, p) satisfy these equations with

$$A(u, p) = -\frac{\partial}{\partial x} p^T B u$$

and

$$u = B^T p.$$

Geometrically, the pair (x, p) is to be thought of as a point in the cotangent bundle T^*X. In this setup each geodesic through x is generated by a choice of $p(0) \in T^*_x X$ but, just as in Riemannian geometry where one does not know *a priori* which values of $\dot{x}(0)$ generate paths over $[0, 1]$ without cut points, here we are not sure *a priori* which values of $p(0)$ generate curves which are free of cut points on $[0, 1]$.

In order to prevent one from attempting to attach intrinsic meaning to an accidental choice of coordinates it is worthwhile to recast these ideas in coordinate free and, while we are at it, global terms. Let X be a manifold and let $\tilde{E}$ be a rank m euclidean vector bundle over X. Let $B: \tilde{E} \to E \subset TX$ be a vector bundle isomorphism. If $\langle \ , \ \rangle$ is the inner product on $\tilde{E}$ then the subbundle of TX defined by E has an inner product which comes from $\langle \ , \ \rangle$. Associated with E is a sequence of derived distributions. Define E^0_x as span $B(x)$ and continue inductively

$$E^{(1)} = \overset{\text{span}}{=} (E^{(0)}_x + [E^{(0)}_x, E^{(0)}_x]), \; E^{(2)} = \overset{\text{span}}{=} (E^{(1)} + [E^{(1)}, E^{(1)}]), \ldots, \text{etc.},$$

where the brackets indicate vector fields which arise as Lie brackets of vector fields in the space indicated. If the dimensions of $E_x^{(i)}$ are, for each i, independent of x then E defines a sequence of derived vector bundles $E^{(0)} \subset E^{(1)} \subset E^{(2)} \subset \cdots$. The condition for the system to be controllable is that this sequence should terminate at TX. Of course E determines, canonically, a dimension m subbundle $E^{\dagger} \subset T^*X$, $E^{\dagger} = \{p \mid p \text{ vanishes on } E\}$. The map $B\colon \tilde{E} \to E$ and the inner product define a map from $T^*X/E^{\dagger}$ into $\tilde{E}$ which is given in coordinates by $p \mapsto B^T p = u$. The pair of equations given above then define a section of the tangent bundle of T^*X. If the controllability condition is satisfied then we get a metric $\rho(\cdot,\cdot)$ on X and we may be sure that any two points in X are joined by a geodesic.

We also point out the following additional result which plays a role later. Suppose that $E^{(1)}$ equals TX. In that case the inner product structure on E can be used to define an inner product on $([E, E] + E)/E$. The idea is analogous to the one whereby an inner product on the space of one forms is used to define an inner product the space of two forms, etc. This goes as follows. Let $b_1, b_2, \ldots, b_m$ be an orthonormal basis for E in some neighborhood $U \subset X$. Any point in $([E, E] + E)/E$ can be then expressed as

$$X = \sum \alpha_{ij}[b_i, b_j] + E.$$

Such a representation is not unique, but among all such representations there is a unique one which minimizes

$$\left(\sum_{ij=1}^{m} \alpha_{ij}^2\right)^{1/2} = \eta(X).$$

This then gives a mapping from $([E, E] + E)/E$ into $\mathbb{R}^{m(m-1)/2}$. It is easily seen to be linear. We define the length of a point in $([E, E] + E)/E$ as the minimum value of $\eta(X)$. It is easy to verify that this defines a norm and that the norm satisfies the parallelogram identity and so it comes from an inner product. Finally, one can check that the norm is independent of the choice of orthonormal basis.

Geodesic Equations

In order to better understand the qualitative behavior of the solutions of the optimal control problem which we introduced in the second section, we now describe a transformation which may be thought of as a partial inverse Legendre transformation. The effect of this transformation is to introduce as many second order equations as possible. Everything here is local.

Given the control equations $\dot{x} = B(x)u$, we then have a subbundle $E = \text{span } B$ in TX. In a neighborhood of any point x_0 we can find an integrable subbundle $\hat{E}$ of TX which is tangent to E at x_0. In local coordinates this amounts to saying that we can write the given equations as

$$\dot{x}_u = B_u u$$
$$\dot{x}_l = B_l u$$

with B_u being an m by m nonsingular matrix and $B_l(x_0) = 0$. For each choice of integrable subbundle $\hat{E}$ tangent to E at x_0 we get such a decomposition of the equations of motion by letting x_u be such that

$$\frac{\partial}{\partial x_u^1}, \ldots, \frac{\partial}{\partial x_u^m}$$

span the integrable subbundle. As noted, E also determines a subbundle $E^\dagger \subset T^*X$, namely the subbundle of one forms which vanish on E. Denoting a typical point in $E^\dagger$ by p_l we can write the equations of the previous section as

$$\begin{aligned}
\dot{x}_u &= B_u u \\
\dot{x}_l &= B_l u \\
\dot{p}_u &= A_{uu}(u, p_u) + A_{ul}(u, p_l) \\
\dot{p}_l &= A_{lu}(u, p_u) + A_l(u, p_l).
\end{aligned}$$

Differentiating the equation $\dot{x}_u = B_u B_u^T p_u + B_u B_l^T p_l$ with respect to time and using the differential equations for p we get a second order equation in x_u. By using $\dot{x}_u = B_u u$ to eliminate u we then end up with a pair of equations of the form

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \Lambda^i_{jk}\dot{x}^j p^k = 0 \qquad x^i \in \{x^1, x^2, \ldots, x^n\}$$
$$\dot{p}^i + F^i_{jk}\dot{x}^j\dot{x}^k + E^i_{jk}\dot{x}^j p^k = 0 \qquad p^i \in \{p^{m+1}, p^{m+2}, \ldots, p^n\}$$

where the coefficients depend on x but not $\dot{x}$ or p. These equations have to be integrated along with the nonholonomic constraints represented by $\dot{x}_l = B_l B_u^{-1}\dot{x}_u$. The symmetries are as follows: Γ^i_{jk} is symmetric in jk and Λ^i_{jk} is skew symmetric in ij.

Since we did not give a canonical way to choose $\hat{E}$ we cannot attach an intrinsic meaning to any aspect of these equations which is not invariant with respect to that choice. However, given any such choice, B_u defines an inner product on E and hence B_u defines a Riemannian structure on the submanifold passing through x_0 and defined by $\hat{E}$. When we change $\hat{E}$ or x_0 we change this Riemannian structure. We call the original system *reducible* if there exists a choice for $\hat{E}$ such that when we write $\dot{x}_u = B_u u$, B_u is of the form

$$B_u(x_u, x_l) = B_0(x_u)\theta(x_u, x_l)$$

with θ orthogonal. Under this circumstance the Riemannian structure does not vary from leaf to leaf and we can recast the entire problem in the following terms. Given an m-dimensional Riemannian manifold M find the shortest path between two points m_0 and m_1 subject to $n - m$ constraints of the form $y_i(1) = \alpha_i$ where y^i satisfies

$$\dot{y}^i = c^i_j(x, y)u^j.$$

In this case Γ^i_{jk} are just the ordinary Christoffel symbols for this Riemannian manifold.

There exists an entire hierarchy of examples of reducible systems, classified on the basis of the properties of the Riemannian space. For example, the space might be taken to be flat, symmetric, etc. Our prototype problem of section two can be restated as the problem of finding the shortest path between two points $\mathbb{R}^2$ such that the area enclosed by the straight line between the two points and the path has a specified value. This family of special cases is therefore related to the isoperimetric problems in the calculus of variations, and in particular, to the problem of Pappus [8], solved by him more than two thousand years ago.

A Local Canonical Form

Just as the local features of Riemannian geometry are greatly clarified by coordinatizing the manifold by Riemann's normal coordinates, in the present context the local features of the optimal control problem may be revealed by an appropriate choice of coordinates. Specifically, we consider $\dot{x} = B(x)u$ under the replacements

$$x \mapsto x = \Psi(x)$$
$$u \mapsto u = \Theta(x)u,$$

where Ψ is a diffeomorphism and $\Theta(x)$ is an orthogonal matrix. This is the natural group to study because of the role of $\langle u, u\rangle = \langle \Theta u, \Theta u\rangle$ in the optimal control problem. What we will find is that it is possible, under a suitable hypothesis, to get an interesting and useful canonical form with respect to this group of transformations. Everything we do here is local.

To begin with we consider $\dim X = 3$ $\dim \tilde{E} = 2$. What we want to establish is that in this case we can arrange matters so that in a neighborhood of x_0 we have

$$\dot{x}^1 = u^1 + r_1$$
$$\dot{x}^2 = u^2 + r_2$$
$$\dot{x}^3 = u^1x^2 - u^2x^1 + r^3$$

where r_1 and r_2 have vanishing first partials and r^3 has vanishing first partials with respect to x^3 and vanishing first and second partials with respect to x^1 and x^2, all at x_0. Moreover, and this is what justifies the particular choices, any other choice of coordinates which enjoys the same properties is related to the given one by a change of variables whose Jacobian at x_0 has the form

$$\left.\frac{\partial\psi}{\partial x}\right|_0 = \left[\begin{array}{c|c} J_u & 0 \\ \hline 0 & J_l \end{array}\right].$$

Therefore we have an intrinsic definition of a direction $(\partial/\partial x^3)$ which, together with Range B, defines a splitting of the tangent bundle. The prototype problem shows us that this is the direction along which the conjugate points of an associated approximating problem emanate from x_0.

To begin the proof of these assertions consider a system

$$\dot{x}^i = u^i + \gamma^i_{jk} x^j u^k; \qquad i, j, k = 1, 2, \ldots, m.$$

As is well known, any m by m by m array such as γ^i_{jk} can be expressed as a sum $q^i_{jk} + \omega^i_{jk}$ with $q^i_{jk} = q^i_{kj}$ and $\omega^i_{jk} = -\omega^k_{ji}$. By changing coordinates according to

$$x^i \mapsto x^i - \tfrac{1}{2} q^i_{jk} x^j x^k$$

$$u^i \mapsto \theta^i_j u^j,$$

where $\theta = \exp(\Omega(x))$ and $\Omega(x) = (\omega^i_{jk} x^j)$, we arrive at a system

$$\dot{x}^i = u^i + r^i$$

for which all the first partials of r^i vanish at 0. Let's call this a "type one" reduction.

We now consider the x_l equations. For notational reasons we write x and y instead of x_u and x_l. Consider then

$$\dot{x}^i = u^i, \qquad i = 1, 2$$

$$y^i = \alpha^i_{jk} y^j u^k + \beta^i_{jk} x^j u^k + q^i_j u^j$$

where q^i_j have vanishing first partials with respect to x and y at zero.

Split β^i_{jk} up as $\beta^i_{jk} = \bar{\beta}^i_{jk} + \hat{\beta}^i_{jk}$ with the former being symmetric with respect to jk and the latter skew symmetric with respect to the same indices. Notice that if we replace y by $y^i - \frac{1}{2}\bar{\beta}^i_{jk} x^j x^k - \alpha^i_{jk} y^j x^k$ then

$$\dot{y}^i = \hat{\beta}^i_{jk} x^j u^k + \hat{q}^i_j u^j$$

where q^i_j have vanishing first partials with respect to x and y at zero. Let's call this a "type two" reduction.

Using a type one reduction followed by a type two reduction we can arrange matters so that the dim $X = 3$, dim $\tilde{E} = 2$ system looks, in a neighborhood of $x = 0$, like

$$\dot{x}^1 = u^1 + r^1$$

$$\dot{x}^2 = u^2 + r^2$$

$$\dot{y} = u^1 x^2 - u^2 x^1 + q^1(x^1, x^2)u^1 + q^2(x^1, x^2)u^2 + r^3$$

where r^3 has the property mentioned above: its first and second partials with respect to x^1 and x^2 vanish at zero, and its first partial with respect to y vanishes at zero. We need to eliminate the quadratic terms q^1 and q^2. To this end substitute

$$y \mapsto y - h(x)$$

with $h(x)$ cubic in x and selected in such a way as to put the expression for $\dot{y}$ in the form

$$\dot{y} = u^1 x^2 - x^2 u^1 + \alpha x^1 y u^1 + \beta x^2 y u^2.$$

It is easy to see that such an h exists. After the further substitution

$$y \mapsto y - \alpha x^1 y - \beta x^2 y$$

we have

$$\begin{aligned} \dot{x}^1 &= u^1 + r^1 \\ \dot{x}^2 &= u^2 + r^2 \\ \dot{y} &= u^1(x^2 + \alpha x^1 x^2 + \alpha y) - u^2(x^1 - \beta x^1 x^2 + \beta y) + r^3. \end{aligned}$$

The final reduction to the canonical form is now effected by the substitutions

$$\begin{aligned} x^1 &\mapsto x^1 + \alpha y + \alpha x^1 x^2 + 2\beta(x^2)^2 \\ x^2 &\mapsto x^2 + \beta y - \beta x^1 x^2 - 2\alpha(x^1)^2 \\ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &\mapsto \exp\begin{bmatrix} 0 & \alpha x^2 - \beta x^1 \\ -\alpha x^2 + \beta x^1 & 0 \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}. \end{aligned}$$

The statement about the form of the Jacobian at zero may be verified by noticing that the linear transformation which defines the above transformation on (x^1, x^2) has no z component if and only if the system is initially in the desired canonical form.

Based on these techniques we can prove the following theorem.

Theorem 1. *Given $\dot{x} = B(x)u$ with* dim $u = m$ *and* dim $x = m(m+1)/2$ *and given that $E^{(1)}$ spans $\mathbb{R}^{m(m+1)/2}$ we can choose coordinates $(x^1, x^2, \ldots, x^m, y^{1,2}, y^{1,3}, \ldots, y^{m-1,m})$ in a neighborhood of a given point, say $x = 0$, so that the equations take the form*

$$\begin{aligned} \dot{x}^i &= u^i + r^i, \qquad i = 1, 2, \ldots, m \\ \dot{y}^{ij} &= u^i x^j - u^j x^i + r^{ij}, \qquad i, j = 1, 2, \ldots, m, \quad i < j \end{aligned}$$

where the r^i and r^{ij} have vanishing first partials with respect to x and y and in addition r^{ij} has vanishing second partials with respect to x^i and x^j. Moreover, given any second set of coordinates with this property it follows that the Jacobian of the diffeomorphism which relates them has the block diagonal form

$$J = \begin{bmatrix} J_{xx} & 0 \\ 0 & J_{yy} \end{bmatrix}$$

when evaluated at zero.

In the next section we will analyze in detail the system defined by this canonical form without the remainder terms. This analysis will explain, in part, the hypothesis that $n = m(m+1)/2$. In fact, even the approximating problem may display a certain lack of robustness with respect to the location of the conjugate points unless this condition on the dimension is satisfied.

Model Spaces

Based on the claim of the previous theorem, the systems of the form

$$\dot{x}^i = u^i, \qquad i = 1, 2, \ldots, m$$
$$\dot{y}^{ij} = u^i x^j - u^j x^i, \qquad i, j = 1, 2, \ldots, m$$

assume a special importance. In this section we explore the associated geodesics. What we will show is that the optimal solution has a remarkably simple structure. One might even think of this class of systems as being the appropriate analog of the flat Riemannian spaces in the present context.

There are many possible notational schemes: we begin with one which is control theoretic and mention at the end an alternative based on differential forms. Consider $x \in \mathbb{R}^m$ and $Y \in o(m)$, the set of n by n skew symmetric matrices. The control system is

$$\dot{x} = u$$
$$\dot{Y} = xu^T - ux^T.$$

It is easy to see that this system is controllable on $\mathbb{R}^n \times o(n)$ and that this is equivalent to the problem defined above. If we are to minimize

$$\eta = \int_0^1 \langle u, u \rangle \, dt$$

subject to fixed boundary conditions $x(0) = 0$, $x(1) = s$, $Y(0) = 0$; if $Y(1) = S$, then an elementary Lagrange multiplier argument shows that there exists a skew symmetric matrix Λ such that

$$\dot{u} + \Lambda u = 0.$$

Thus

$$\dot{x} = u = e^{\Lambda t} a,$$

and the corresponding value of η is just $\|a\|$. Since $x(0) = 0$ and $Y(0) = 0$ we can also write

$$x(t) = e^{\Lambda t} b - b$$

and

$$Y(1) = \int_0^1 ((e^{\Lambda t} b - b) b' \Lambda' e^{\Lambda' t} - e^{\Lambda t} \Lambda b (b' - b' e^{\Lambda' t})) \, dt.$$

Sign error

The optimal trajectories to points on the set $x(1) = 0$ are especially interesting. In this case the expression for Y simplifies to

$$Y(1) = \int_0^1 e^{\Lambda t}(bb'\Lambda' - \Lambda bb')e^{\Lambda' t}\, dt.$$

Theorem 2. *The control $u(\cdot)$ defined on $[0, 1]$ which minimizes*

$$\eta = \int_0^1 \langle u, u\rangle\, dt$$

for

$$\dot{x} = u; \qquad x(0) = 0 \qquad x(1) = 0$$
$$\dot{Y} = xu^T - ux^T; \qquad Y(0) = 0 \qquad Y(1) = Y$$

satisfies

$$\dot{u} + \Lambda u = 0$$

for some skew-symmetric matrix Λ. The associated value of ρ is given by

$$\rho((0, 0), (0, Y)) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + r/2\lambda_r$$

where $\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_r$ are the eigenvalues of Y listed in decreasing order according to the size of the imaginary part. Any two optimal controls u_1 and u_2 transferring $(0, 0)$ to (x, Y) are related by $u_1 = \theta u_2$ for some orthogonal matrix θ such that $\theta Y\theta' = Y$. The point (x, Y) is conjugate to the point $(0, 0)$ if, and only if, x belongs to an invariant subspace of Y which is not in the complement of Ker Y.

Proof. The first observation is that the range space of the operator

$$W = \int_0^1 e^{\Lambda t}bb'e^{\Lambda' t}\, dt$$

is the same as that of $(b, \Lambda b, \ldots, \Lambda^{m-1}b)$ and that the dimension of this range space is upperbounded by the number of distinct eigenvalues of Λ. Second, if $e^{\Lambda}b = b$ then Λ must have at least rank W eigenvalues of the form $2\phi\pi i$ with ϕ an integer since b is necessarily a linear combination of eigenvectors corresponding to such eigenvalues. Finally, if θ is any orthogonal matrix we have $\rho((0, 0), (0, Y)) = \rho((0, 0), (0, \theta' Y\theta))$. We may, therefore, understand the general situation by understanding the case where Λ is of the form

$$\Lambda = \begin{bmatrix} 0 & \omega_1 & & & \\ -\omega_1 & 0 & & & \\ & & 0 & \omega_2 & \\ & & -\omega_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

with $\omega_k = 2\pi\phi_k$ and no ϕ_k is repeated. In this case a calculation shows that

$$W\Lambda' - \Lambda W' = \begin{bmatrix} 0 & (b_1^2 - b_2^2) & 0 & 0 & \cdots \\ (-b_1^2 - b_2^2) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & (b_3^2 - b_4^2) & \cdots \\ 0 & 0 & -(b_3^2 - b_4^2) & 0 & \cdots \\ & & \cdots & \cdots & \end{bmatrix}.$$

But this makes the solution obvious. Since the cost is $\|\Lambda b\|$ we need to pick $\{\phi_1, \phi_2, \ldots, \phi_{m/2}\}$ to be $\{1, 2, \ldots, m/2\}$ if m is even and $\{0, 1, \ldots, (m-1)/2\}$ if m is odd. The total cost will then be expressible in terms of eigenvalues of Y. Say that the eigenvalues of Y with positive imaginary parts are $i\lambda_1, i\lambda_2, \ldots, i\lambda_r$ listed in decreasing size of the imaginary part. Since the eigenvalues of $W\Lambda - \Lambda' W$ are $(b_1^2 - b_2^2)/\phi_1$, etc., we see that the minimum cost is just

$$\eta^* = \lambda_1 + 2\lambda_2 + 3\lambda_3, \ldots, r\lambda_n, \qquad r \le m/2.$$

As for the lack of uniqueness of u, of course $\dot{x} = u$ implies $\theta\dot{x} = \theta u$ and so θu and u accomplish the same transfer as long as $\theta' Y \theta = Y$. On the other hand, in view of the specific form of the optimal control we see that any two optimal u's which steer $(0, 0)$ to (x, Y) must be so related.

It is worth remarking that while $\rho((0, 0), (0, \alpha Y)) = |\alpha|\, \rho((0, 0), (0, Y))$ it does not define a norm on the space of skew symmetric matrices because the unit ball is not convex. The geometry of the unit ball is partly explained by the remark that the line segments in its boundary correspond to certain line segments in a Cartan subalgebra of $So(n)$.

If we consider a more general version of this problem whereby we wish to control

$$\dot{x} = u$$

$$\dot{y}^i = x^T \Omega_i u, \qquad i = 1, 2, \ldots, r$$

with the Ω_1 skew-symmetric, then it is no longer true that the conjugate points have such a nice structure; in fact, even the connected subset which is conjugate to $(0, 0)$ and contains $(0, 0)$ need not admit the structure of a manifold in any natural way.

There are two possible generalizations of this problem which are interesting and have been investigated in special cases. The first concerns the possibility that it is not $E^{(1)}$ which spans TX but rather some higher $E^{(i)}$. The second concerns isoperimetric problems which are not based on flat spaces but rather spaces of constant curvature.

Let $\Lambda(\mathbb{R}^m)$ denote the 2^m-dimensional Grassmann algebra. Recall that $\Lambda(\mathbb{R}^m)$ splits as the sum of $m+1$ vector spaces $\Lambda^0(\mathbb{R}^m) + \Lambda^1(\mathbb{R}^m) + \cdots + \Lambda^m(\mathbb{R}^m)$, the pth of which is of dimension $\binom{m}{p}$; $\Lambda^p(\mathbb{R}^m)$ is called the space of p-forms. There is an antisymmetric multiplication in $\Lambda(\mathbb{R}^m)$ denoted by $\wedge$ and called exterior multiplications; it respects the above decomposition in that

$$\wedge : \Lambda^p(\mathbb{R}^m) \times \Lambda^q(\mathbb{R}^m) \to \Lambda^{p+q}(\mathbb{R}^m).$$

Now consider a control system for which $u \in \Lambda^1(\mathbb{R}^n)$, $x \in \Lambda(\mathbb{R}^n)$ and

$$\dot{x}_0 = 0$$
$$\dot{x}_1 = u \wedge x_0$$
$$\cdots\cdots$$
$$\dot{x}_p = u \wedge x_{p-1}.$$

If we set $x_0 = 1$ and delete the first equation, this is a system for which $E^{(i)} = \Lambda^1(\mathbb{R}^m) + \cdots + \Lambda^i(\mathbb{R}^m)$. Above we considered the special case $p = 2$.

A second kind of generalization which yields interesting results concerns systems for which X is principle bundle over a Riemannian space M, u takes on values in TM and the equations of motion are of the form

$$\dot{m} = u$$
$$\dot{Y} = \sum u^i \Omega^i(m) Y$$

where Y is some representation of the group, the $\Omega^i(m)$ belong to the appropriate Lie algebra. The special case of an S^1 bundle over S^2 has been investigated, by J. Baillieul [5] and N. Gunther and T. Goodwille [unpublished]. □

A Second Order Operator

Considerations having to do with stochastic differential equations containing m-dimensional Wiener processes lead, under an appropriate hypothesis, to a naturally defined second order partial differential operator associated with our basic problem. The resulting operator is a generalization of the Laplace–Beltrami operator; it will be hypoelliptic but typically not elliptic.

Recall that an m-dimensional Wiener process w has a $0(m)$ invariance in the sense that the statistical properties of the solution of an Itô equation

$$dx = f(x)\,dt + G(x)\,dw$$

are identical with those of the Itô equations

$$dx = f(x)\,dt + G(x)\theta(x)\,dw$$

where $\theta(x)$ is an orthogonal matrix depending smoothly on x. This $0(m)$ invariance means that the same group of transformations investigated in connection with the local canonical form is relevant here as well.

Given $\dot{x} = B(x)u$ as in the second section we define

$$f^i(x) = -\frac{1}{2}\frac{\partial b^i_j}{\partial x_k} b^k_j \quad \text{(summation convention)}$$

and consider the stochastic equation in the sense of Itô.

$$dx = f(x)\,dt + B(x)\,dw.$$

Together with this equation we consider the equation for the evolution of the associated probability density function $\rho(t, x)$ which is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x^i} f^i(x)\rho + \frac{1}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} b^i_k b^j_k \rho$$

or

$$\frac{\partial \rho}{\partial t} = L_+ \rho.$$

The operator L_+ is not, however, invariantly defined because the probability density ρ is the density with respect to the measure $dx_1, dx_2, \ldots, dx_n$ and, when we change coordinates in X, this transforms by multiplication by the determinant of the Jacobian. The effect on L_+ is therefore

$$L_+ \mapsto \psi^{-1} L_+ \psi$$

where ψ is the determinant of the Jacobian.

If the underlying manifold has a Riemannian structure on it then it has a natural measure, namely $\sqrt{\det G}\; dx_1\, dx_2 \ldots dx_n$ where G is the metric tensor. In that case it may be verified that the operator defined by

$$\frac{1}{(\det G)^{1/2}} L_+ (\det G)^{1/2}$$

is the standard Laplace–Beltrami operator. Thus to get an invariantly defined operator in the present context it is enough to single out a set of diffeomorphisms which are related by transformations whose Jacobians are constant.

Based on the work we have done we are in a position to identify a suitable subset of the set of all diffeomorphisms in the following case. Suppose that for $\dot{x} = B(x)u$ we have $E^{(1)} = TX$ and suppose that $\dim X = m(m+1)/2$ so that Theorem 1 applies. We have a splitting of the tangent space at each point, $T_x X = E_x + F_x$. We also have a method of constructing an inner product on $([E, E] + E)/E$. However, in view of the given decomposition of T_x it can be naturally identified with $E \oplus ([E, E] + E)/E$. Since both these factors have euclidean structures we have obtained from the euclidean structure on E a euclidean structure on $T_x X$. What role this might have in the study of the original problem remains to be investigated, however it does let us define a volume form on $T_x X$ and hence singles out an invariant second order operator.

As remarked at the end of the second section, the second order operator

$$L_+ = \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)^2$$

plays the role of the heat operator on the metric space defined by $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = xv - yu$. A calculation shows that it is also the second order operator defined by the above procedure. It is hypoelliptic but not elliptic.

In view of the many interesting results which relate the magnitude of the eigenvalues of a Laplace–Beltrami operator to the lengths of geodesics on a compact Riemannian manifold it is natural to expect that this would be a fruitful area of study in the present context. In an earlier paper [9] the spectrum of the Fokker–Planck operator was calculated for a class of problems which fit our framework and the spectrum was related, in a rough way, to the controllability of the systems. The time it takes a Fokker–Planck equation to reach a steady state is of some interest in applications and this is related to the spectrum of the Fokker–Planck equation; perhaps the time is right for a more general study of this type.

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