ASYMPTOTIC STABILITY AND FEEDBACK STABILIZATION

R. W. Brockett

Abstract. We consider the local behavior of control problems described by ($\dot{x} = dx/dt$)

$$\dot{x} = f(x,u) \qquad ; \qquad f(x_o,0) = 0$$

and more specifically, the question of determining when there exists a smooth function $u(x)$ such that $x = x_o$ is an equilibrium point which is asymptotically stable. Our main results are formulated in Theorems 1 and 2 below. Whereas it might have been suspected that controllability would insure the existence of a stabilizing control law, Theorem 1 uses a degree-theoretic argument to show this is far from being the case. The positive result of Theorem 2 can be thought of as providing an application of high gain feedback in a nonlinear setting.

1. Introduction

In this paper we establish general theorems which are strong enough to imply, among other things, that

a) there is a continuous control law $(u,v) = (u(x,y,z),v(x,y,z))$ which makes the origin asymptotically stable for

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= v\\ \dot{z} &= xy\end{aligned}$$

and that

b) there exists no continuous control law $(u,v) = (u(x,y,z), v(x,y,z))$ which makes the origin asymptotically stable for

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = yu - xv$$

The first of these implies that the null solution of Euler's angular velocity equations can be made asymptotically stable with two control torques aligned with principle axes. (See [1,2] for a general discussion, but not this particular result.) The second provides a counter example to the oft repeated conjecture asserting that a reasonable form of local controllability implies the existence of a stabilizing control law. Section 2 gives certain background material in control theory. In section 3 we formulate our nonexistence result and in section 4 we give a criterion for the existence of stabilizing control laws.

Sussmann [3] gives an example of a system in $\mathbb{R}^2$ which is controllable in a strong sense and yet fails to have a continuous feedback control law yielding global asymptotic stability. His example involves both bounds on the controls and nonlocal considerations, ours involves neither.

2. Control Systems

We intend to work locally in this paper, but even so it is perhaps worthwhile to make a few remarks about a global formulation. A more detailed and systematic account can be found in [3], but in any case the reader familiar with control theory can go directly to section 3.

Let X be a differentiable manifold and let $\pi : E \to X$ be a vector bundle over X. Let TX denote the tangent bundle of X and let π^*TX denote the pullback of TX over E. A section of π^*TX is then an assignment of a velocity vector in TX for each point in E. If we choose a local trivialization of E and pick coordinates then a section $\gamma \in \Gamma(E,\pi^*TX)$ is equivalent (in an obvious notation) to

$$\dot{x} = f(x,u)$$

such a γ is called a control system.

A section $\alpha \in \Gamma(X,E)$ is an assignment of a pair (u,α) corresponding to each x and so locally it is given by a function $\alpha(x)$. We denote by γ^α the section of $\Gamma(E,\pi^*TX)$ defined in coordinates by