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On the Synthesis of Compliant Mechanisms

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Abstract

Although it has been traditional in the study of mechanisms to separate kinematic issues from elastic ones, there is a significant set of problems whose solutions require the application of techniques from both fields. In this paper, through an emphasis on a geometrical interpretation of these subjects, we formulate and solve a number of questions which arise in the design of compliant devices. An important part of the paper is the development of a certain extension of the Frenet-Serret equations which describes the elasto-geometric properties of a thin beam viewed as a non zero cross-sectional extension of a space curve.

1 Introduction

The subject of kinematics plays an important role in robotics as does the study of compliance. Often these are treated as independent disiplines but, because of certain practical issues that arise in robotics, some unification is desirable. In this paper we consider a class of problems involving both kinematics in the sense of references [1] [2] and elastic behavior in the sense of [3]. The areas of potential application include the design of mechanisms for hard and soft automation, including remote compliance devices, as well as the more traditional areas of application associated with the design of mechanisms.

The literature on these matters has coalesced in a variety of places. On one hand there has been various treatments of mechanical systems based on bond graphs, of which we may mention the recent paper of Breedweld [4]. The paper of Anderson and Spong [5] adopts a more mathematical style but has the same general goal. Loncaric [6] [7] has examined a number of fundamental ideas in compliance synthesis, developing a geometric approach along the lines of reference [8]. The analysis of concrete problems, such as those found in Whitney [9] and Asada and Kakumoto [10] has, however, usually been viewed as the main source of motivation for this type of work. Standing somewhat apart from the applications literature are the more mathematical approachs to these problem areas as exemplified by Marsden and Hughes [11] and Arnold [12]. Our motivation comes from desire to merge and extend aspects of classical kinematic design with aspects of the design of elastic systems so as to make available suitable tools for the synthesis of assemblies and mechanisms .

The paper is organized as follows. In section 2 we discuss the geometric structure associated with the combination of elastic members and holonomic rigid body kinematics. The key idea here is to make explicit a basic methodology which allows one to extend the earlier geometrical treatments in such a way as to cover aspects of elasto-kinematics not resolved by earlier workers. In section 3 we discuss the interconnection of elasto-kinematic systems. The basic building blocks available for the synthesis of elasto-kinematical systems are nonelastic kinematic chains and simple elastic two ports. In section 4 we work out a description of the latter which is suitable for our purposes. We develop an infintessimal method for characterizing the spatial compliance of a quasi one-dimensional continuum, expressing the end to end compliance as a function of the geometric and elastic properties along the member. The last section is devoted to examples.

2 Elasto-kinematics

In physical terms our starting point is this. We assume that there is a base structure, which we may imagine to be fixed in space, and that relative to this base structure there are certain ports. At each of these ports there is attached to the base structure one rigid body. This rigid body may be partly constrained by the base structure so as to give it between one and six degrees of freedom, relative to the base structure. Levers, shafts, switches, and joysticks, etc. should come to mind. We assume that there are springs, gears, levers, etc. inside the base structure and that these may provide some relationships among the variables at the ports. Our goal is to derive the general properties of such systems and to do this in a setting that does not require linearization.

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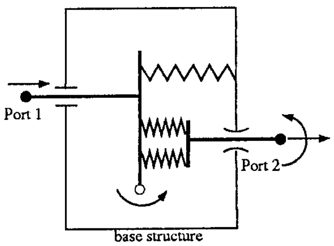

Figure 1: The ports

We assume that the configuration space relative to the base structure for the rigid body at the i^{th} port can be identified with a differentiable manifold $\mathcal{Y}_i$, the dimension of $\mathcal{Y}_i$ being the number of degrees of freedom associated with the i^{th} port when the rigid bodies at all the other ports are completely unconstrained. We define $\mathcal{Y}$ as the product manifold $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \cdots \times \mathcal{Y}_m$. Clearly each configuration of the port variables can be identified with a point in the manifold $\mathcal{Y}$ although not every point in $\mathcal{Y}$ will necessarily arise as a configuration because there may be constraints between behavior at port i and behavior at port j. Indeed, one of our goals is to describe the ways in which these relationships can manifest themselves.

With this goal in mind, we recall a few ideas from differential geometry. If M is an n-dimensional differentiable manifold then in a neighborhood of a point $m_o \in M$ there is a map from M into $\mathbb{R}^n$ given by

$$m \mapsto (q_1(m), q_2(m), \cdots q_n(m))$$

which provides local coordinates for M near m_0

If $m(\cdot)$ is a parametized path on M that passes through m_0 say $m(0) = m_0$ then dm/dt can, at $t = 0$ be thought of as defining a point in $\mathbb{R}^n$, namely the point $(dq_1/dt, dq_2/dt, \cdots dq_n/dt)$. We denote by $T_{m_0}M$ the n-dimensional vector space of possible values of the derivative of curves in M at the point m_0. It is called the tangent plane at m_0. The union over all m_0 of the tangent planes of M is denoted by TM and is called the **tangent bundle**. It can be thought of as a differentiable manifold of dimension $2n$.

There are many different notations in common use for the representation of linear functionals, including "dot" as in $f \cdot x$, the inner product notation $< f, x >$, the notation $f[x]$, etc. If v belongs to a vector space V we denote by V^* the set of all linear functionals on V and use the notation $f[v]$ to denote the value of the linear functional f evaluated on v. The set of all linear functionals on $T_{m_0}M$ will be denoted by $T^*_{m_0}M$ and the union of the $T^*_{m_0}M$ over all m_0 by T^*M ; it is called the **cotangent bundle**. Like the tangent bundle it can be thought of as a differentiable manifold of dimension $2n$.

It is now a more or less standard idea in geometrical mechanics to think of the force/torque vector and the configuration variable as being a point in a single space, namely the cotangent bundle of the configuration space. (See, for example, Marsden and Hughes [11]). One pictures T^*M as being obtained from M by attaching the n-dimensional vector space consisting of the possible force/torque vectors to each point of M. The vectors in this vector space are to be thought of as being linear functionals on the tangent space. Thinking of T_mM as the vector space of possible velocities associated with trajectories in M that pass through the point m, the expression for the work done by a force acting on a moving particle

$$w = \int_a^b f\,[\dot{m}]\,dt$$

gives a physical meaning to the process whereby we identify forces with linear functionals on the velocity.

Suppose that we map a neighborhood of a point in M to a neighborhood of zero in $\mathbb{R}^n$ with $(q_1, q_2, \cdots q_n)$ being a representation for the point $\mathbb{R}^n$. In a similar way $(q_1, q_2, \cdots, q_n,\ p_1, p_2, \cdots, p_n)$ is to represent a point in T^*M. We may take $(p_1, p_2, \cdots p_n)$ to represent a point in T^*_qM. By a q-dependent change of basis in that space we can arrange matters so that the linear functionals $p_1, p_2, \cdots p_n$, when evaluated on components of the velocity vector $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, satisfy the equation

$$p_i\,[\dot{q}_j] = \delta_{ij}$$

with δ_{ij} being the Kronecker symbol.

There is a distinguished bilinear form ω defined on $T_{m_0}M \oplus T^*_{m_0}M$ which we may specify in terms of the special coordinates just described

$$\omega((\dot{q}_a, p_a), (\dot{q}_b, p_b)) = p_a[\dot{q}_b] - p_b[\dot{q}_a]$$

In terms of a choice of coordinates for T^*M obeying the above rule, any n-dimensional submanifold L of T^*M that has the property that for each path $q(\cdot) \in M \cap L$ any two pairs $(\dot{q}_a, p_a)$ and $(\dot{q}_b, p_b)$ satisfy:

$$p_b\,[\dot{q}_a] - p_a\,[\dot{q}_b] = 0$$

will be said to be a **Lagrangian submanifold.**

The main idea is that if $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_m$ is the configuration space of an elasto-kinematic system then $T^*\mathcal{Y}$ can be thought of as the space in which the forces (including torques) and displacements coexist. If the interconnections within the system arise through reciprocal springs (i.e. spring forces derivable from a potential) and holonomic kinematics, then the set of force position pairs which actually occur will lie in a Lagrangian submanifold of $T^*\mathcal{Y}$. In symbols, we may say that a reciprocal holonomic elasto-mechanical system defines a Lagrangian submanifold $L \subset T^*M$ and that each such submanifold defines such a system. This statement has appeared in the literature in varying degrees of generality, see e.g. Loncaric [7] and Marsden and Hughes [11]. Here we want to pursue certain aspects of this idea which arise when one does not assume that elastic effects are pervasive but, instead, assumes that the kinematic and elastic effects are on a more or less even footing. (Compare with Loncaric [7]).

Our first goal is to bring out the role of two integers that, differentiate between the number of elastic degrees of freedom and the number of kinematic degrees of freedom.

The intuitive idea is this. If $T^*\mathcal{Y}$ is the set of force-displacement pairs for an elasto-kinematic system then, at a given configuration y, it will happen that a virtual displacement anywhere in a certain r-dimensional subspace of $T_y\mathcal{Y}$ will produce no change in the force. There will also be a subset of $T_y^*\mathcal{Y}$ in which an applied force/torque will produce no displacement. Geometrically speaking, let L be the Lagrangian manifold of the system. At each $y \in \mathcal{Y}$, $T_y\mathcal{Y}$ intersects the tangent space to the Lagrangian manifold TL in a ν-dimensional space. That is $\nu(y) = \dim\,(T_yL\ \cap T_y\mathcal{Y})$. Likewise L intersects $T_y^*\mathcal{Y}$ in a $\mu(y)$ dimensional space.

We can express these ideas in physical terms by saying that there is a ν-dimensional space of velocities that do no work and a μ-dimensional space of forces that do no work.

These remarks suggest the form of the constraint equations. At each point $y \in \mathcal{Y}$ there are relations that constrain the forces and velocities. These can be organized by constructing an n by $2n$ dimensional matrix [A,B] having the property that

$$[A(y), B(y)]\begin{bmatrix} \dot{q} \\ p \end{bmatrix} = 0$$

Theorem 1: If $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is given by

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

and if $Ax + Bf = 0$ defines a Lagrangian submanifold of $\mathbb{R}^{2n}$ relative to J, then by means of a change of coordinates of the form $x \mapsto\ q = Tx$; $f \mapsto p = (T^T)^{-1}f$ one can arrange matters so that the Lagrangian subspace is described by

$$\begin{bmatrix} I_\mu & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & I_\nu \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0$$

with Q and H invertible and $Q^{-1}H$ symmetric.

Proof: If A is invertible then $Ax + Bf = 0$ implies that $x + A^{-1}Bf = 0$. If any two solutions, say $x_1 + A^{-1}Bf_1 = 0$ and $x_2 + A^{-1}Bf_2 = 0$ are to have the property that $< x_1, f_2 > - < x_2, f_1 >= 0$ we see that $A^{-1}B - B^T(A^{-1})^T = 0$ and so $A^{-1}B$ is symmetric. If A is not invertible we can choose nonsingular matrices R and S so that

$$RAS = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

If x is to be replaced by $q = S^{-1}x$ then we must replace f by $p = S^T f$ in order to perserve the symplectic form J. This means that we can re-express matters as

$$\tilde{A}q + \tilde{B}p = 0$$

with

$$\tilde{B} = RB(S^{-1})^T = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

If $\tilde{A}q_1 + \tilde{B}p_1 = 0$ and $\tilde{A}q_2 + \tilde{B}p_2 = 0$ is to imply that $< q_1, p_2 > - < q_2, p_1 >= 0$ then, since the last part of q_1 and q_2 is unconstrained, we see that the last part of p must be zero. Thus it is necessary that $B_{21} = 0$ and that B_{22} is invertible. Given that B_{22} is invertible, and given the structure of $\tilde{A}$, it is not hard to see that we can take B_{22} to be I_ν for some integer ν. Moreover, by choice of a suitable pre multiplication matrix we can combine rows so as to make $B_{12} = 0$. We have, then

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q_u \\ q_l \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ 0 & I_\nu \end{bmatrix}\begin{bmatrix} p_u \\ p_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For the final steps we can ignore q_l and p_l. As we have seen, B_{11} is symmetric. If its rank is r then by an orthogonal transformation $B_{11} \mapsto \Theta B_{11}\Theta^T$ we can put B_{11} in the form

$$\Theta B_{11}\Theta^T = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{B} \end{bmatrix}$$

from which the theorem follows.

At the i^{th} port there is a force-configuration space $T^*\mathcal{Y}_i$ and if we impose zero force at all the other ports we will be able to split the tangent space of $\mathcal{Y}_i$ into two parts according to the criterion that in the first part motion requires no force where as in the second it does. We write this as

$$T_{y_i}\,\mathcal{Y}_i\ =\ V_{y_i}\ +\ W_{y_i}$$

By analogy with electrical network theory we can think of V_{yi} as the subspace in which the open circuit impedance is zero. Likewise, if we freeze the positions at all ports except the i^{th} port then we can split the force space

$$T^*_{yi}\,\mathcal{Y}_i\ =\ \tilde{V}_{yi}\ +\ \tilde{W}_{yi}$$

with $\tilde{W}_{yi}$ being the space in which an arbitrary force produces no displacement.

Theorem 2: Let $\mathcal{Y}$ be an m dimension configuration manifold for an input-output elastomechanical system. Let L be the Lagrangian submanifold of $T^*\mathcal{Y}$ that describes the generalized force-displacement relation. Let $\nu(y)$ and $\mu(y)$ of theorem 1 be independent of y. Then any realization of the elastomechanic system requires at least $n - \nu - \mu$ scalar spings.

This is just a physical interpretation of the structure revealed by theorem 1.

Definition: We will say that a system is **purely kinematic** if $\dim(y) = \nu(y) + \mu(y)$ at each point $y \in \mathcal{Y}$. It is said to be **purely elastic** if $\nu(y) = \mu(y) = 0$

3 Interconnection

One of the reasons for introducing ports and describing the systems without reference to internal variables is so that interactions between different systems of this type can be easily understood. In this section we want to model the interaction process.

Let S_1 and S_2 be elasto-kinematic systems with configuration spaces $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_m$ and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_p$. There are many ways in which interaction between these systems might occur but we limit the discussion to rigid interconnections between variables at port $\mathcal{Y}_i$ and variables at port $\mathcal{X}_i$.

Definition: Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be given by $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}, \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_s$ and $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_t$. We will say that the elasto-kinematic system $L_z \subset T^*Z$ is the result of an **interconnection** of the elasto-kinematic system $L_x \subset T^*X$ and the elasto-kinematic system $L_y \in T^*\mathcal{Y}$ if for some ordering of the factors

$$\mathcal{X}_i = \mathcal{Y}_i \ ; \ i = 1, 2 \cdots, \alpha$$

and $\mathcal{Z} = \mathcal{X}_{\alpha+1} \times \cdots \times \mathcal{X}_r \times \mathcal{Y}_{\alpha+1} \times \cdots \times \mathcal{Y}_s$ with the port variables at $\mathcal{X}_i$ $\mathcal{Y}_i$ being equated.

4 Stiffness of a curved beam element

Our point of view here is that it is useful to think of a general elasto-kinematic synthesis problem as being separable into units which are then interconnected. The interconnection itself will require that one take into account both spatial relations and force/torque relations. Typically the spatial relations will consist of a nominal relative positioning of the parts followed by an analysis of the effects of incremental deformations. In order to bring this kind of analysis within the scope of our theory we give here a geometrical analysis of the stiffness properties of a common connector, namely a thin curved beam of variable cross section. Our development includes both the "in the large" geometry of the beam and the incremental stiffness analysis in a unified way.

The notation follows that of Simo [14]. A thin rod can be described geometrically by a curve $\Phi(s)$ in $\mathbf{R}^3$, which passes through the centroid of the rod, together with an orthogonal basis $\mathbf{E}(s)$ which is parameterized by the arc length along the curve s. The tangent vector to the centroidal curve at s_0 is denoted $\mathbf{E_3}(s_0)$, where

$$\mathbf{E_3}(s) = \frac{\frac{\partial \Phi(s)}{\partial s}}{\|\frac{\partial \Phi(s)}{\partial s}\|}$$

The cross-section associated with point s_0 on the arc is a compact subset of the plane spanned by the orthonormal vectors $\mathbf{E_1}(s_0)$ and $\mathbf{E_2}(s_0)$ To describe the evolution of these vectors as a function of arc length, let

$$\Lambda_0(s)^T = \begin{pmatrix} \mathbf{E_1}^T(s) \\ \mathbf{E_2}^T(s) \\ \mathbf{E_3}^T(s) \end{pmatrix}$$

The skew-symmetric matrix $\Omega_0 = \frac{\partial \Lambda_0{}^T(s) \Lambda_0(s)}{\partial s}$ is taken to be :

$$\Omega_0 = \begin{pmatrix} 0 & K_3 & -K_2 \\ -K_3 & 0 & K_1 \\ K_2 & -K_1 & 0 \end{pmatrix}$$

Following deformation, the line of centroids is denoted $\phi(s)$. Consider a basis $\mathbf{e_i}$ adapted to the deformed element. The basis vectors $\mathbf{e_1}(s_0)$ and $\mathbf{e_2}(s_0)$ remain in the plane of the original cross-section. Given this choice, $\mathbf{e_3}(s_0)$ is determined. Because of shearing strains, the vector tangent to $\phi(s)$ at s_0, denoted $\mathbf{t}(s_0)$, is not necessarily parallel to $\mathbf{e_3}(s_0)$. The basis vectors $\mathbf{E_i}$ and $\mathbf{e_i}$ are related by the equation:

$$\mathbf{e_i}(s) = \Lambda(s)\mathbf{E_i}$$

The skew-symmetric matrix $\Omega = \Lambda^T(s)\frac{\partial \Lambda(s)}{\partial s}$ is taken to be:

$$\Omega = \begin{pmatrix} 0 & \kappa_3 & -\kappa_2 \\ -\kappa_3 & 0 & \kappa_1 \\ \kappa_2 & -\kappa_1 & 0 \end{pmatrix}$$

We define the shearing and extensional strain measure by the expression:

$$\Gamma = \Lambda_0^T \Lambda^T \frac{\partial \phi(s)}{\partial s} - \mathbf{I}_3$$

and the bending strain $\omega = (\kappa_1, \kappa_2, \kappa_3)$, which is related to the skew-symmetric matrix Ω in the standard way (see [8]). We will use the notation $[\omega] = \Omega$

The equations for balance of forces and torques in the static case are:

$$\frac{\partial \mathbf{f}}{\partial s} + \bar{\mathbf{q}} = 0$$

and:

$$\frac{\partial \mathbf{m}}{\partial s} + \frac{\partial \phi}{\partial s} \times \mathbf{f} + \bar{\mathbf{m}} = 0$$

where $\bar{\mathbf{m}}$ and $\bar{\mathbf{f}}$ are the applied moment and linear force per unit arclength.

By making the assumption that the beam is thin compared to the radius of curvature, we can establish the approximate relations between the strain and the force components. These approximations are:

$$\mathbf{f} = \mathrm{diag}(GA, GA, EA)\Gamma$$
$$\mathbf{m} = \mathrm{diag}(I_1 E, I_2 E, GJ)\mathbf{K}$$

where GJ is the torsional stiffness, EI_i is the bending stiffness about the i axis, GA is the shear stiffness, and EA is the axial stiffness.

It is of interest to determine how the nominal curvature of the beam together with the bending and twisting stiffnesses can be used as controls to generate a specific driving-point compliance at the tip of the beam. For this purpose consider "growing" a curved beam, ignoring shear and axial elongation.

At the point s the stiffness relationship is:

$$K(s)x(s) = f(s)$$

The generalized force vector $f(s) = (f, m)$ has components of linear force and torque expressed in terms of the E_i basis. (undeformed beam-following) basis and the displacement vector

$$x = (x_1, x_2, x_3, \theta_1, \theta_2, \theta_3)$$

where

$$\phi(s) = \Phi(s) + x_i E_i(s)$$

and

$$\theta_i(s) = \int_0^s \kappa_i(\sigma)\, d\sigma$$

We consider only the first term in the expansion for

$$\Lambda(s) = \mathbf{I} + \int_0^s \Omega(\sigma)\, d\sigma$$

First we give some definitions. The relationship:

$$\begin{pmatrix} f(0) \\ m(0) \end{pmatrix} = T^{-1} \begin{pmatrix} f(s) \\ m(s) \end{pmatrix}$$

where:

$$T = \begin{pmatrix} \Lambda_0(s) & 0 \\ [\Psi(s)]\Lambda_0(s) & \Lambda_0(s) \end{pmatrix}$$

gives the force at the beam base due to a tip force with components given with respect to the moving reference frame at s_0

The matrix T evolves according to:

$$\dot{T} = T \begin{pmatrix} \Omega_0 & 0 \\ [(0,0,1)] & \Omega_0 \end{pmatrix}$$

or

$$\dot{T} = TA$$

Theorem 3: The stiffness as a function of arc length satisfies the differential equation:

$$\dot{K}(s) = K(s)A^T(s) + A(s)K(s) + K(s)BK(s)$$

where A is identified above and:

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(I_1E, I_2E, JG)^{-1} \end{pmatrix}$$

For the planar case, we investigate a segment with curvature given as a function of arclength: $k(s)$. In this case it is possible to explicitly write:

$$\Phi(s_0) = \int_0^{s_0} \begin{pmatrix} cos\alpha(s)ds \\ sin\alpha(s)ds \end{pmatrix}$$

where

$$\alpha(s) = \int_0^s k(\sigma)d\sigma$$

Let:

$$\Lambda_0(s) = \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$$

$$\alpha(s) = \int_0^S k(\sigma)d\sigma$$

and:

$$\Lambda(s) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$\theta(s) = \int_0^S \kappa(\sigma)d\sigma$$

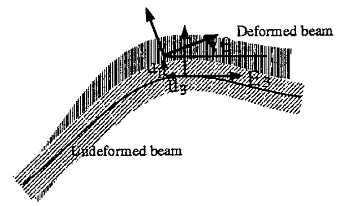

Figure 2: Curved Planar Beam

To explicitly formulate the compliance relations it is necessary to choose coordinates. We choose to express the components of force and displacement with respect to the basis vectors E_i of the undeformed structure. Given:

$$\phi(s) = \Phi(s) + u_1(s)E_1(s) + u_3(s)E_3(s)$$

and

$$f = f_1E_1(s) + f_3E_3(s)$$

together with the simplified constitutive relations, the infinitessimal compliance relations are:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_3 \\ \dot{\theta} \\ \dot{f}_1 \\ \dot{f}_3 \\ \dot{m} \end{pmatrix} = \begin{pmatrix} 0 & K & -1 & \frac{1}{GJ} & 0 & 0 \\ -K & 0 & 0 & 0 & \frac{1}{EA} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{IE} \\ 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ \theta \\ f_1 \\ f_3 \\ m \end{pmatrix}$$

5 Example: A Planar End-Effector

In descriptions of end-effectors, the center of compliance has been shown to be of importance. Consider the design of the wrist as shown in Fig. 3. The two curved beams are reflection symmetric. By altering the curvature only (the bending stiffness is constant) it is possible to alter the ratio of horizontal to vertical stiffness as well as the center of compliance. Consider two curves for which $EI = 10^3$ and the total arc length is 2.5. Let K_{xx} be the stiffness in the x direction, and $K_{x\theta}$ be the term which couples rotation and horizontal motion, and κ the nominal curvature of the structure.

In case 1 we have:

$$\begin{aligned} k(s) &= -0.1 * \sin(1.25 * s) \\ K_{xx} &= 1987.85 \\ K_{x\theta} &= -1632.15 - 5988.c + 1987.85d \end{aligned}$$

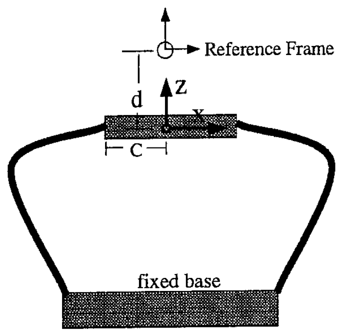

Figure 3: A compliant wrist

$$
\begin{aligned}
K_{zz} &= 76598.8 \\
K_{\theta\theta} &= 3277.49 - 6489.5c + 76598.8c^2 \\
&- 3264.3d - 11976.cd + 1987.85d^2
\end{aligned}
$$

In case two we are able to get:

$$
\begin{aligned}
k(s) &= -0.4\sin(1.25 * s) \\
&+ 1.4\sin(2.51 * s) - 2.5\sin(3.76 * s) \\
K_{xx} &= 3763.81 \\
K_{x\theta} &= 609.5 - 7539.9c + 3763.81d \\
K_{zz} &= 33620.5 \\
K_{\theta\theta} &= 4834.2 - 19514.8c + 33620.5c^2 \\
&+ 1219d - 15079.7cd + 3763.81d^2
\end{aligned}
$$

By a specific choice of the distance between the tips of the curved beam tips (c), it is readily possible to choose an offset (d) for the center of compliance. It is also shown that the ratio of vertical to horizontal stiffness is readily controlled.

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