A Geometrical Formulation of the Dynamical Equations Describing Kinematic Chains

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1 Introduction

Although textbooks often regard the problem of generating the dynamical equations of a kinematic chain as being just an exercise in classical mechanics, the complexity one encounters in this process has lead to repeated attempts to organize the computations in a efficient and transparent way. The literature has not stabilized on any particular scheme, however, and one is tempted to guess that this is because the attempts at simplification usually involve the use of a number of ad hoc definitions and choices of notation. In this paper we show that certain standard ideas from geometry, especially the idea of a one parameter group of transformations, when used systematically, lead to a reasonably elegant general formulation of the dynamics of open chain manipulators. We build on ideas which have been applied to manipulators kinematics by Brockett [3] and McCarthy [5], for example, and to the description of mechanical compliance by Loncarcic [4].

In this paper we develop a general expression for the kinetic energy associated with a kinematic chain and use it to derive the dynamical equations. In our expression for the kinetic energy, the dependence on the chain's parameters is particularly transparent. Such a representation is desireable in applications such as adaptive control and robot calibration, where one needs to isolate distinct physical quantities. The kinetic energy is expressed using standard geometric operations e.g., group multiplication, exponentiation, and adjoint mappings. We take advantage of Lie-theoretic identities to simplify the expressions for those derivatives of the inertia matrix which appear in Lagrange's equations. For example, an elegant expression for the Coriolis terms is provided. The equations of motion for a serial chain are written in a general form which requires no adaptation for specific problems. Finally, we use this representation to classify dynamically balanced chains.

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2 Geometric Formalism

2.1 Background

We begin with a few definitions and an explanation of our notation. Denote euclidean 3-space by E^3. The special Euclidean group is the group of rigid transformations on E^3, i.e., transformations of the form $x \mapsto Rx + p$ where R is an orientation preserving orthogonal transformation and p is in E^3. The group of such transformations will be denoted by SE(3). The group of orientation-preserving orthogonal transformations is called the special orthogonal group and will be denoted by SO(3). Given a reference frame, a transformation may be represented by the 4×4 matrix $T = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix}$. The path of a rigid body in euclidean space corresponds to a path in SE(3) in the following way. Let a be a fixed reference frame, and b be a frame attached to the body. At time t, the position of the origin of b with respect to a is given by $p_{ab}(t)$, and the columns of $R_{ab}(t)$ are the unit vectors of the axes of of frame b with respect to a. The definition $T_{ab} = \begin{pmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{pmatrix}$ then implies a convenient rule for composition of transformations: $T_{ab}T_{bc} = T_{ac}$.

We recall that a Lie group G is a smooth manifold on which is defined continuous group operations of multiplication and inversion. The Lie group structure of SE(3) is utilized extensively in this paper. The Lie algebra $\mathfrak{g}$ associated with G is a vector space together

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with a bilinear map $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, such that (i) $[x, y] = -[y, x]$ (anti-symmetry) and (ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity).

We represent elements of SE(3) by 4×4 matrices of the type described above. It is sufficient for the purposes of this paper to think of Lie groups and algebras as consisting of square matrices. The matrix representation of the algebra $\mathfrak{g}$ can then be found by differentiating one-parameter paths on the group at the identity. The Lie bracket is defined as the matrix commutator: $[A, B] = AB - BA$, where A,B are in $\mathfrak{g}$. For matrix representations of the group, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is simply

$$\mathrm{Ad}_g\, \eta = g \eta g^{-1} \tag{1}$$

for $g \in G$ and $\eta \in \mathfrak{g}$ both matrices. The map $\mathrm{ad}_\eta : \mathfrak{g} \to \mathfrak{g}$ is given by:

$$\mathrm{ad}_\eta\, \nu = [\eta, \nu]$$

where $\eta, \nu \in \mathfrak{g}$. The following identities for Ad and ad will be used frequently:

$$\mathrm{Ad}_a\, \mathrm{Ad}_b = \mathrm{Ad}_{ab}$$

and

$$\mathrm{Ad}_a(\mathrm{ad}_\eta\, \nu) = \mathrm{ad}_{\mathrm{Ad}_a \eta}(\mathrm{Ad}_a\, \nu)$$

They can be verified directly by the definitions. Finally, since we are dealing with matrices, the exponential mapping $\exp: \mathfrak{g} \to G$ is just the familiar matrix exponential: $\exp(A) = I + A + \frac{1}{2}A^2 + \ldots$ for $A \in \mathfrak{g}$.

The 4×4 matrix representation of the algebra $\mathfrak{se}(3)$ associated with SE(3) is readily seen to consist of all matrices of the form: $\begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix}$ where Ω is a 3×3 skew-symmetric matrix and $v \in R^3$. Velocity vectors associated with paths in SE(3) are identified with the algebra in two canonical ways (Abraham and Marsden, 1978). These two identifications have an appealing physical interpretation. Let the 4×4 matrix $T_{ab}(t)$ describe the path of body b relative to frame a. $\frac{d}{dt}T_{ab}\,|_{t_0}$ is a tangent vector at $T_{ab}(t_0) \in$ SE(3) which is mapped into the matrix representation of the algebra by multiplication on the left by the inverse of T_{ab}:

$$T_{ab}^{-1}\frac{d}{dt}T_{ab}(t_0) \stackrel{\text{def}}{=} \begin{pmatrix} \Omega_{ab} & v_{ab} \\ 0 & 0 \end{pmatrix} \stackrel{\text{def}}{=} V_{ab}$$

$V_{ab} \in \mathfrak{se}(3)$ is often referred to as the "body-fixed" generalized velocity of body b because it is independent of the position of the fixed reference frame a. The second form, $\frac{d}{dt}T_{ab}T_{ab}^{-1}$, is commonly called the "spatial" velocity of the rigid body.

3 Rigid Body Dynamics

In this section, we will explore the dynamics of a rigid body in terms of the 4×4 matrices described above. Of course, the standard approach in this case is to write the equations in terms of the center of mass/ principal axes and separate equations for conservation of linear and rotational momentum. The complexity encountered in multi-link chain dynamics, however, makes a general 4×4 matrix formulation of rigid body dynamics a useful building block.

3.1 Kinetic Energy

We first will write the equations of motion for a rigid body with a frame at the center of mass, aligned with the principal axes. Let $V \in \mathfrak{se}(3)$ be the "body-fixed" velocity with respect to this frame. frame on the body. Straight-forward calculation yields that the kinetic energy of the body may be expressed as

$$T = \frac{1}{2}(\mathrm{tr}(VMV^T) + \mathrm{tr}(JVJV^T)) \tag{2}$$

with M and J 4×4 matrices defined as

$$J = \sqrt{\det(I)} \begin{pmatrix} I^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} \mathrm{Diag}(0,0,0) & 0 \\ 0 & m \end{pmatrix}$$

Here, $I = \mathrm{Diag}(I_1, I_2, I_3)$, and m, I_i are the mass and principal moments of inertia of the rigid body.

The kinetic energy of course can be expressed in terms of another frame, which we index by p. The "body-fixed" velocity of this frame is V_p, $V_p = \mathrm{Ad}_{T_{pc}}\, V$, where T_{pc} is the transformation from the center of mass frame c to frame p. The kinetic energy:

$$T = \frac{1}{2}(\mathrm{tr}(G_p V_p M_p V_p^T) + \mathrm{tr}(J_p V_p J_p V_p^T))$$

where:

$$G_p = T_{cp}^T T_{cp}$$

$$M_p = T_{pc} M T_{pc}^T$$

$$J_p = T_{pc} J T_{pc}^T$$

This kinetic energy defines an inner product on $T\,$SE(3):

$$\langle V_p, V_p \rangle = \mathrm{tr}(G_p V_p M_p V_p^T) + \mathrm{tr}(J_p V_p J_p V_p^T)$$

3.2 Equations of Motion

Lagrangian dynamics are derived by requiring that the first variation of the integral:

$$L = \int \langle V_p, V_p \rangle$$

be zero. For any $W \in se(3)$, the trajectory $V_p, \dot{V}_p$ of the rigid body must then satisfy:

$$\langle \dot{V}_p, W \rangle = \langle V_p, [W, V_p] \rangle \qquad (3)$$

This form of the equations will be useful for deriving general "euler-angle" equations of motion, and for the multi-link chain.

As an alternative to Eq. 3, a matrix-form of the equations of motion takes the form

$$G_p \dot{V}_p M_p + J_p \dot{V}_p J_p = (\bar{I} G_p)^T [J_p V_p J_p, V_p^T]$$

where $\bar{I} = \begin{pmatrix} \mathrm{Diag}(1,1,1) & 0 \\ 0 & 0 \end{pmatrix}$.

We now show how Eq. 3 can be used to equations of motion in six coordinates. Let $x = (x_1, x_2, \ldots x_6)$ be coordinates of the second kind for SE(3). That is, the position/orientation of frame p with respect to an inertial frame 0 is

$$T_{0p} = \exp(A_1 x_1) \exp(A_2 x_2) \ldots \exp(A_6 x_6)$$

Then:

$$V_p = \sum_{i=1}^{6} \mathrm{Ad}_{T_{6i}} A_i \dot{x}_i$$

for $T_{k6} = \exp(A_k x_k) \exp(A_2 x_2) \ldots \exp(A_6 x_6)$ and:

$$\begin{aligned} \dot{V}_p &= \sum_{i=1}^{6} \mathrm{Ad}_{T_{6i}} A_i \ddot{x}_i + \\ & \sum_{i=1}^{6} \sum_{j=i+1}^{6} [\mathrm{Ad}_{T_{6i}} A_i, \mathrm{Ad}_{T_{6j}} A_j] \dot{x}_i \dot{x}_j \end{aligned}$$

Letting $W = \mathrm{Ad}_{T_{6k}} A_k$ the kth equation of motion is:

$$\sum_{i=1}^{6} g_{ki} \ddot{x}_i + \sum_{i=1}^{6} \sum_{j=1}^{6} \gamma_{ijk} \dot{x}_i \dot{x}_j = 0$$

where:

$$\begin{aligned} g_{ki} &= \langle \mathrm{Ad}_{T_{6i}} A_i, \mathrm{Ad}_{T_{6k}} A_k \rangle \\ \gamma_{ijk} &= \frac{1}{2}(\langle [\mathrm{Ad}_{T_{6k}} A_k, \mathrm{Ad}_{T_{6j}} A_j], \mathrm{Ad}_{T_{6i}} A_i \rangle + \\ & \langle [\mathrm{Ad}_{T_{6k}} A_k, \mathrm{Ad}_{T_{6i}} A_i], \mathrm{Ad}_{T_{6j}} A_j \rangle + \\ & \langle [\mathrm{Ad}_{T_{6n}} A_n, \mathrm{Ad}_{T_{6m}} A_m], \mathrm{Ad}_{T_{6k}} A_k \rangle \end{aligned}$$

and $m = \max(i,j)$, $n = \min(i,j)$. This formula, while not as simple as the usual rigid body equations of motion, generalizes nicely to the multi-link chain.

4 Serial Linkages and the Product of Exponentials Formula

The serial linkage to be considered in this paper is shown in Fig. 1.

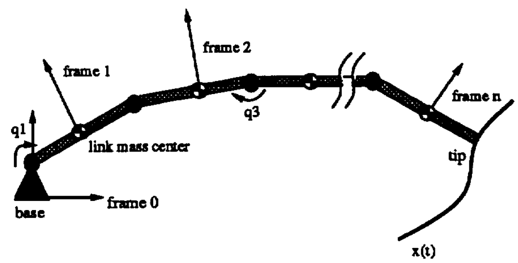

Figure 1: Serial Mechanism

Recall that R_{0k}, p_{0k} are the orientation and position of frame k relative to frame 0. Frame k is at the center of mass of link k with axes corresponding to the principle axes of the link. It has been observed that $T_{0k} = \begin{pmatrix} R_{0k} & p_{0k} \\ 0 & 1 \end{pmatrix}$ can be expressed as the product of one-parameter subgroups of SE(3), i.e. group elements of the form $\exp(\theta B)$ where θ is a parameter and B is a constant element of the algebra. The expression:

$$T_{0k}(q) = \exp(q_1 A_1) \exp(q_2 A_2) \ldots \exp(q_k A_k) N_k \qquad (4)$$

is often called the product of exponentials formula for the kinematic chain. The vector $q = (q_1, q_2, \ldots q_n)^T$ represents the joint configuration. $A_i \in se(3), N_k \in$ SE(3) are constant. For a physical interpretation of A_i in terms of twists, see for example Paden and Sastry [6]. N_k is the position/orientation of frame k with respect to frame 0 at $q = 0$.

The velocity in frame i-fixed coordinates of link i is given by:

$$T_{0i}^{-1} \dot{T}_{0i} = N_i^{-1} (\sum_{k=1}^{i} P_{k,i}^{-1} A_k P_{k,i} \dot{q}_k) N_i = V_{0i} \qquad (5)$$

where:

$$P_{k,i} = \prod_{j=k+1}^{i} \exp^{(q_j A_j)}$$

To minimize the number of subscripts, we will frequently make use of the notation $\mathrm{Ad}_{P_{ij}} = \mathrm{Ad}_{ij}$. It follows that:

$$V_{0i} = \mathrm{Ad}_{N_i^{-1}} (\sum_{k=1}^{i} \mathrm{Ad}_{ik} A_k \dot{q}_k) \qquad (6)$$

4.1 Lagrange's Equations

The kinetic energy T of the kinematic chain is a sum over the kinetic energies of the individual links, $T = \sum_{i=1}^{n} T_i$, where $T_i = \langle V_{0i}, V_{0i} \rangle_i$.

$$T_i = \frac{1}{2}(\mathrm{tr}(V_{0i} M^i V_{0i}^T) + \mathrm{tr}(J^i V_{0i} J^i V_{0i}^T)) \qquad (7)$$

M^i, J^i are defined as M and J in the above section for the mass and inertia properties of link i. In order to write the kinetic energy in terms of the joint velocities, Eqs. 6 and 7 are combined.

$$T_i = \sum_{k=1}^{i} \sum_{m=1}^{i} \langle \mathrm{Ad}_{ik} A_k, \mathrm{Ad}_{im} A_m \rangle_i \dot{q}_m \dot{q}_k$$

where:

$$\langle \cdot, \cdot \rangle_p = (\mathrm{tr}(\cdot) \bar{M}_p (\cdot)^T \bar{G}_p) + \mathrm{tr}((\cdot) \bar{J}_p (\cdot)^T \bar{J}_p))$$

$$\bar{M}_p = N_p M^p N_p^T$$
$$\bar{J}_p = N_p J^p N_p^T$$
$$\bar{G}_p = (N_p^{-1})^T N_p^{-1}$$

The coefficients g_{ij} of the quadratic form: $T = \sum_{i,j} g_{ij} \dot{q}_i \dot{q}_j$ are simply given by:

$$g_{ij} = \sum_{p=\max(i,j)}^{N} \langle \mathrm{Ad}_{pi} A_i, \mathrm{Ad}_{pj} A_j \rangle_p \qquad (8)$$

We now consider in this framework the equations of motion for a kinematic chain where only kinetic energy contributes to the Lagrangian. Trajectories of the mechanism may then be associated with geodesics on a manifold. When kinetic energy given by the quadratic form: $\frac{1}{2} g_{ij}(q) \dot{q}_i \dot{q}_j$ the Coriolis term for the k^{th} equation of motion (i.e. $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$) is given by

$$\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q_j} + \frac{\partial g_{kj}}{\partial q_i} - \frac{\partial g_{ij}}{\partial q_k}\right) \dot{q}_i \dot{q}_j = \Gamma_{ijk} \dot{q}_i \dot{q}_j$$

Proposition : The Christoffel symbols Γ_{ijk} take the form: $\Gamma_{ijk} = \sum_{p=\max(i,j,k)}^{N} \gamma_{ijk}^p$. The γ_{ijk}^p are symmetric in i, j, and for $m = \max(i, j), n = \min(i, j)$,

$$\begin{aligned} \gamma_{ijk}^p \ = \ & \frac{1}{2}(\langle [\mathrm{Ad}_{pk} A_k, \mathrm{Ad}_{pj} A_j], \mathrm{Ad}_{pi} A_i \rangle_p + \\ & \langle [\mathrm{Ad}_{pk} A_k, \mathrm{Ad}_{pi} A_i], \mathrm{Ad}_{pj} A_j \rangle_p + \\ & \langle [\mathrm{Ad}_{pn} A_n, \mathrm{Ad}_{pm} A_m], \mathrm{Ad}_{pk} A_k \rangle_p \end{aligned}$$

Proof: This fact can be directly shown by repeated application of the identity:

$$\frac{d}{dq_k} \mathrm{Ad}_{pj} A_j = [\mathrm{Ad}_{pj} A_j, \mathrm{Ad}_{pk} A_k]$$

for $j < k < p$, and 0 otherwise.

Using the expressions for Γ_{ijk} and g_{ij}, the equations of motion for the n-link chain are simply

$$g_{kj} \ddot{q}_j = \Gamma_{ijk} \dot{q}_i \dot{q}_j$$

where summation over repeated indices is implied.

5 Dynamically Balanced Mechanisms

Let us agree to say that a kinematic chain is dynamically balanced if the mass matrix (Eq. 8) is constant. Clearly, in this case, the dynamics take the simple form $M\ddot{q} = 0$. In this section, our formulation is used to study the structure of dynamically balanced mechanisms.

Our example involves the design of planar mechanisms with rotational joints for which a general solution is straight-forward. Requiring the diagonal terms in the inertia matrix to be constant is necessary and sufficient. The planar mechanism and notation is shown in Fig. 2 Frame k is located at the center of

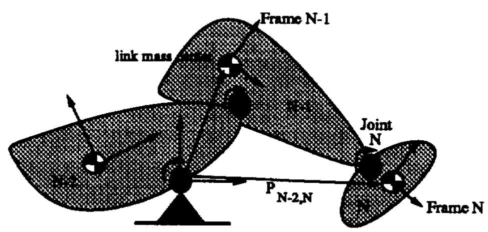

Figure 2: Planar Mechanism

mass of body k, and frame $\bar{k}$ is fixed to body k, with origin corresponding to the location of the kth joint. $p_{\bar{i}k}$ is the distance between joint i and the center of mass of link k, and $R_{\bar{i}k}$ is the 2×2 rotation matrix between frames $\bar{i}$ and k. Following the notation above, $T_{\bar{i}k} = \begin{pmatrix} R_{\bar{i}k} & p_{\bar{i}k} \\ 0 & 1 \end{pmatrix}$.

We define for the planar mechanism:

$$E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_i \end{pmatrix}$$

$$J^i = \begin{pmatrix} I_i & 0 & 0 \\ 0 & I_i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The ith diagonal term of the mass matrix is

$$\begin{aligned} g_{ii} &= \sum_{k=i}^{N} \mathrm{tr}(T_{k\bar{i}} E T_{\bar{i}k} M^k (T_{k\bar{i}} E T_{\bar{i}k})^T) + \\ &\quad \mathrm{tr}(J^k T_{k\bar{i}} E T_{\bar{i}k} J^k (T_{k\bar{i}} E T_{\bar{i}k})^T) \end{aligned}$$

Only the first term on the right hand side of the above equation depends on the joint positions. This term, which we will call d_i is equal to:

$$d_i = \sum_{k=i}^{N} m_k \parallel p_{\bar{i}k} \parallel^2$$

If the mechanism is to have a constant mass matrix, all d_i must be constant. By inspection d_n is always constant, and d_{n-1} is constant only when the center of mass of link n is at joint n. Setting d_{n-2} constant provides a "first moment" condition, namely, that the mass of link $n-1$ and mass of link $n-1$ must be balanced about joint $n-1$, or more specifically

$$m_{n-1} p_{(n\bar{-}1),n-1} = m_n p_{(n\bar{-}1),n}$$

where $p_{(n\bar{-}1),n-1}$ is the position of the center of mass of link $n-1$ with respect to joint $n-1$, and $p_{(n\bar{-}1),n}$ is the position of the center of mass of link n with respect to joint $n-1$. Using back substitution, the requirement for d_{k-1} to be constant is simply:

$$m_k p_{\bar{k},k} = (\sum_{j=k+1}^{n} m_j) p_{\bar{k},k+1}$$

Spatial mechanisms can also be dynamically balanced, and examples are readily constructed.

6 Conclusion

The dynamical problems that one encounters in the study of robotics are often quite complex. For this reason, the examples appearing in the literature are, for the most part, either overly simplified or quite specific. In this paper we have attempted to maintain an appropriate level of generality while avoiding specialized notation. Our efforts can be compared with earlier work applying general mechanical network-like analysis to this class of problems [1] [7], [2].

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