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On explicit steady-state solutions of Fokker-Planck equations for a class of nonlinear feedback systems

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Abstract

We study the question of existence of steady-state probability distributions for systems perturbed by white noise. We describe a class of nonlinear feedback systems for which an explicit formula for the steady-state probability density can be found. These systems include what has been called monotemperaturic systems in earlier work. We also establish relationships between the steady-state probability densities and Liapunov functions for the corresponding deterministic systems.

1 Introduction

The study of linear systems excited by white noise of constant intensity is greatly facilitated by the fact that one has an explicit formula for the solution of the Fokker-Planck equation which describes the evolution of the probability density. For nonlinear systems the situation is quite different. Namely, not only are the transient solutions difficult to find, but even the steady-state solutions are hard to characterize. In this paper we show that for a certain class of nonlinear systems the steady-state densities can be found explicitly. These systems correspond to feedback systems of Lur'e type, and can also be thought of as a generalization of the monotemperaturic systems introduced in [2]. Some related ideas and applications are discussed in [3], [4], and [6].

We consider Itô stochastic systems of the form

$$dx = Axdt + Bdw + kf(c^Tx)dt \tag{1}$$

where $x(t), k, c \in \mathbb{R}^n$, w is an m-dimensional Wiener process, and A and B are matrices of appropriate dimensions. Throughout the paper we make the following two assumptions.

a) All eigenvalues of A have negative real parts.

b) (A,B) is a controllable pair.

The problem under consideration is that of existence of a steady-state probability distribution for the process $x(t)$. Let us first recall what happens in the linear case ($k = 0$). It is well known that the steady-state probability density is

$$\rho(x) = \frac{1}{\sqrt{(2\pi)^n \det Q}}\, e^{-\frac{1}{2}x^TQ^{-1}x}$$

where Q is the positive definite symmetric *steady-state variance matrix* satisfying the equation

$$AQ + QA^T = -BB^T. \tag{2}$$

We will be concerned with extending this result to the nonlinear case. In Section 2 we formulate a condition on the parameters of the system (1) that enables us to obtain an explicit formula for a steady-state probability density. This condition will be called the *compatibility condition*, and systems for which it is satisfied will be called *compatible*. A physical interpretation of compatibility is given in Section 3 in terms of certain concepts from statistical thermodynamics. Namely, systems that are monotemperaturic in the sense of [2] turn out to be compatible. In Section 4 we single out a class of systems for which the compatibility condition takes a particularly transparent form, and show how the steady-state probability densities are related to Liapunov functions for deterministic nonlinear feedback systems. In Section 5 we discuss convergence of the probability distribution associated with (1) to steady state. Finally, in Section 6 we use the work of Zakai on the existence of steady-state probability distributions to obtain bounds on second moments for certain non-compatible systems.

2 Compatible systems

Let us denote by Q the solution of the equation (2). We will say that the system (1) is *compatible* if the vectors k and AQc are proportional, i.e., if the following *compatibility condition* is satisfied:

$$k = \lambda AQc \quad \text{for some } \lambda \in \mathbb{R}. \tag{3}$$

We will look for a steady-state probability density taking the form

$$\rho(x) = Ne^{-\frac{1}{2}x^TQ^{-1}x - \lambda F(c^Tx)} \tag{4}$$

where F is a *potential*: $F(z) = \int_0^z f(v)dv$, and $N > 0$ is a normalization constant.

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Theorem 1 *If the compatibility condition (3) is satisfied, then the function* ρ *given by (4) is a steady-state probability density for the process described by (1) whenever* f *is such that* $\rho \in L^1(\mathbb{R}^n)$.

REMARK 1. The last requirement is not fulfilled automatically. However, ρ does belong to $L^1(\mathbb{R}^n)$ if, for example, $\lambda > 0$ and $xf(x) \geq 0$ for all x.

Proof. The Fokker-Planck operator L corresponding to (1) is defined by the formula

$$L\rho = -(\mathrm{tr}A + \sum_{i=1}^{n} k_i c_i f'(c^T x))\rho + \frac{1}{2}\sum_{j,k=1}^{n}(BB^T)_{jk}\rho_{x_j x_k} - (\sum_{i,j=1}^{n} A_{ij}x_j + \sum_{i=1}^{n} k_i f(c^T x))\rho_{x_i}.$$

Combining the terms and making use of (2) and (3), it is not hard to see that the above expression equals zero when ρ is given by (4). □

To gain some insight into the meaning of the compatibility condition (3), consider the case when f is piecewise constant (as, for example, in systems with quantized measurements). For each region where f is constant we can find a gaussian density that satisfies the equation $L\rho = 0$ everywhere in that region. Then (3) guarantees that these gaussians fit together at the boundaries between such regions to form a continuous steady-state density. We show in [6] how the solutions obtained here can be used to optimize the steady-state performance of stochastic quantized feedback control systems.

We can switch to new coordinates in which $Q = TI$ for some $T > 0$. The structure of compatible systems is then revealed by the following statement.

Corollary 1 *If* $\Omega = -\Omega^T$, *then the system*

$$\begin{aligned} dx &= (\Omega - \frac{1}{2T}BB^T)x dt + B dw \\ &+ \lambda T(\Omega - \frac{1}{2T}BB^T)c f(c^T x) dt \end{aligned} \tag{5}$$

is compatible.

EXAMPLE 1. Consider the system

$$dx_i = f_i(x_1, \ldots, x_n)dt + b dw_i,\ i = 1, \ldots, n \tag{6}$$

where w_i's are independent scalar Wiener processes. We will call such a system *gradient* if there exists a function $\phi(x_1, \ldots, x_n)$ such that $f_i = -\frac{\partial \phi}{\partial x_i}$. It is not difficult to verify that a steady-state density is then given by

$$\rho(x) = Ne^{-2\phi(x)/b^2}$$

(whenever $\rho \in L^1(\mathbb{R}^n)$). In fact, (6) belongs to a general class of systems that take the form

$$\dot{x} = -\nabla\phi(x) + B\dot{w}$$

where the gradient ∇ is computed with respect to the Riemannian metric given by $G = (BB^T)^{-1}$. For a detailed study of such systems, extended also to degenerate diffusions, see [4]. An arbitrary compatible system will possess, in addition to gradient terms, certain "skew-symmetric" terms which do not change the steady-state probability distribution (more precisely, these come from vector fields of divergence zero that are everywhere tangential to the equiprobable surfaces). In fact, all compatible systems naturally fall into the framework of [4] for the case of $\mathbb{R}^n$ with a constant metric (cf. Section 5 below). A special class of such systems in $\mathbb{R}^2$ has been described by Rueda in [7].

EXAMPLE 2. The Newton's second law for a nonlinear spring in a viscous fluid in the presence of random external forces may be expressed by the second-order equation

$$\ddot{x} + a\dot{x} + f(x) = \dot{w} \tag{7}$$

where $\dot{w} = \frac{dw}{dt}$ is white noise and $a > 0$. The total energy of the system is $\frac{1}{2}\dot{x}^2 + F(x)$, and a steady-state probability density is

$$\rho(x, \dot{x}) = Ne^{-2a(\frac{1}{2}\dot{x}^2 + F(x))}. \tag{8}$$

This formula reflects the fact that the levels of equal energy are at the same time the levels of equal probability in steady state, and ensures that the fluctuation introduced by the presence of white noise and the energy dissipation due to the damping term $a\dot{x}$ eventually neutralize each other. See [3] for a generalization of these ideas and an application to function minimization using simulated annealing.

To see how this example fits into the above framework, consider the following auxiliary system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-\epsilon & 1\\ 0 & -a\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\dot{w} + \begin{pmatrix}0\\ -1\end{pmatrix}f(x) \tag{9}$$

where $\epsilon > 0$. One can check that the compatibility condition (3) holds for (9) with the proportionality constant $\lambda = 2(a + \epsilon) \to 2a$ as $\epsilon \to 0$, which reveals the meaning of the constant $2a$ in the formula (8). If we compute the steady-state density for (9) using the formula (4) and then take the limit as $\epsilon \to 0$, we arrive precisely at (8).

The steady-state probability density (8) is in fact unique. This is a consequence of a result by Zakai [8] for a class of systems that includes (7) as a special case (see Theorem 2 below).

3 Compatibility and statistical thermodynamics

We are now in position to give an interpretation of the compatibility property on physical grounds. It involves systems that describe the behavior of electrical networks with noisy resistors in Nyquist-Johnson form. If all the resistors are of the same temperature T, the system is called *monotemperaturic*. This concept was first mathematically defined in [2], where the authors give a canonical representation for such systems in the form

$$\begin{aligned} \dot{x} &= (\Omega - \frac{1}{2T}GG^T)x + G\dot{w} + Du, \\ \dot{y} &= -D^T x - Fu + \sqrt{2TF}\dot{\nu} \end{aligned} \tag{10}$$

Here $\Omega = -\Omega^T$, $F = F^T$, and $\dot{w}$ and $\dot{\nu}$ are independent white noise processes. The steady-state variance for (10) is $Q = TI$, so we can say that in steady state all the modes possess equal energy. This property is sometimes referred to as the *equipartition of energy* property.

In the present framework, certain types of circuits with nonlinear capacitors or inductors are described by equations of the form (1). We claim that by closing the feedback loop in (10) we can obtain a compatible system. Indeed, let $u = y + f(y)$ (assuming single-input, single-output case, otherwise do it for each pair (u_i, y_i)). This yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \Omega - \frac{1}{2T}GG^T & D \\ -D^T & -F \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & \sqrt{2TF} \end{pmatrix} \begin{pmatrix} \dot{w} \\ \dot{\nu} \end{pmatrix} + \begin{pmatrix} D \\ -F \end{pmatrix} f(y). \tag{11}$$

We have thus obtained a system that takes the form (5) described in Corollary 1. Summarizing, we can say that compatibility can be thought of as a natural property of monotemperaturic systems with nonlinear reactances. Notice that (1) is more general than (11) since the noise matrix in (1) is not necessarily block diagonal.

4 Liapunov functions

Consider a system excited by scalar white noise

$$\dot{x} = Ax + b\dot{w} + kf(c^T x).$$

Since by the assumptions made in the Introduction (A, b) is a controllable pair, in the appropriate basis the linear part of the system takes the standard controllable companion form. Moreover, if k and b are proportional, then we can multiply f by a scalar if necessary and arrive at

$$p(D)x + f(c(D)x) = \dot{w}. \tag{12}$$

Here $D = \frac{d}{dt}$, and $p(D) = D^n + p_{n-1}D^{n-1} + \cdots + p_1 D + p_0$ and $c(D) = c_{n-1}D^{n-1} + \cdots + c_1 D + c_0$ are polynomials. The class of systems thus constructed includes (7), and is of considerable interest despite the special form of (12) (see, e.g., [8]).

In this section we will be concerned with formulating conditions on the polynomials p and c under which (12) is compatible. We will adopt certain results from [1] regarding the Lur'e problem of absolute stability for the deterministic counterpart of (12)

$$p(D)x + f(c(D)x) = 0. \tag{13}$$

We will assume that $xf(x) > 0$ for all x except $x = 0$, and that either the equation $p(D)x = 0$ is asymptotically stable or $p(D) = Dh(D)$ with $h(D)x = 0$ asymptotically stable. Denoting $Dc(D)$ by $m(D)$, assume also that the function $\frac{m(s)}{p(s)}$ is positive real. Then we can apply the classical *factorization lemma* to conclude that there exists a unique polynomial $r(s)$ with real positive coefficients and no zeros in the right half-plane, such that $\mathrm{Ev}p(s)m(-s) = r(s)r(-s) = r^+(s)r^-(s)$ (here Ev stands for the even part of a polynomial). We construct a Liapunov function for (13) as follows:

$$V(x) = \int_0^x [p(D)x\, m(D)x - (r^-(D)x)^2]dt + F(c(D)x) \tag{14}$$

where $\frac{\partial F(c(D)x)}{\partial x^{(i)}} = c_i f(c(D)x)$ as before. In [1], the problem of absolute stability for (13) is investigated with the aid of the function (14). It can be shown that V is well defined, positive definite, and that its derivative along the solutions of (13) is given by

$$\dot{V} = -(r^-(D)x)^2. \tag{15}$$

Obtaining the Liapunov function (14) is a matter of multiplying both sides of (13) by $m(D)x$, integrating by parts, and completing a square if necessary. In many situations (cf. Example 2 of Section 2) such a Liapunov function arises naturally as the total energy of the system.

Now, given a polynomial $p(D)$, let us choose $c(D)$ by setting each c_i to p_{i+1} if i is even and to 0 if i is odd. Notice that $m(D)$ is then simply the odd part of $p(D)$, and therefore $\mathrm{Ev}p(D)m(-D) = m(D)m(-D)$, so we see that $\frac{m(s)}{p(s)}$ is positive real and $r^-(D) = m(-D)$. We can also take $m(D)$ to be a constant multiple of $\mathrm{Odd}p(D)$ as in (7), which amounts to a simple modification of the nonlinearity. We will now use the Liapunov function (14) to arrive at a steady-state density for (12).

Theorem 2 *Suppose that either $p(D)x = 0$ is asymptotically stable or $p(D) = Dh(D)$ with $h(D)x = 0$ asymptotically stable, and that*

$$c(D) = \frac{1}{aD}\mathrm{Odd}\, p(D) \tag{16}$$

with $a > 0$. Then the function

$$\rho(x) = Ne^{-2aV(x)} \tag{17}$$

is a unique steady-state probability density associated with the system (12).

Proof. Let us consider the case when n is even. The equation for a steady-state probability density can be written as

$$\dot{\rho} = p_{n-1}\rho + \frac{1}{2}\frac{\partial^2 \rho}{\partial x_{n-1}^2} \tag{18}$$

where $\dot{\rho}$ stands for the derivative of $\rho(x)$ along the solutions of the deterministic system (13). Using (15) and (17) we can write (18) as

$$2a(r^{-}(D)x)^2 = p_{n-1} - ac_{n-2} + 2a^2(c_{n-2}x_{n-1} + \cdots + c_0 x_1)^2 = 2(a\,m(D)x)^2$$

and this is true by hypothesis.

The case of odd n is treated similarly. The uniqueness of the steady-state probability density (17) follows from the work of Zakai [8, Theorems 3 and 4]. □

From the results of Section 2 it follows that if we want (12) to be compatible, the choice of $c(D)$ is unique up to a constant.

5 Convergence to steady state

It is well known that the time-varying solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = L\rho \tag{20}$$

takes the form

$$\rho(t,x) = \sum_{i=0}^{+\infty} p_i(t) e^{\lambda_i t} g_i(x)$$

where λ_i's are the eigenvalues of L, p_i's are polynomials in t, and g_i's are functions of x. Thus, eigenvalues of L provide information about the convergence of the stochastic process to steady state. In the paper by Holley et al. [5], and more recently in [3] and [4], Fokker-Planck operators and their spectral properties were studied with the view towards applications to function minimization.

In our previous notation, consider the function

$$\phi = \frac{1}{4}x^T Q^{-1} x + \frac{1}{2}\lambda F(c^T x)$$

(cf. Example 1). Define the vector $\nabla\phi$ by

$$(\nabla\phi)_i = \sum_{j=1}^{n} (BB^T)_{ij}\phi_{x_j}.$$

In view of (2) and (3) we have

$$\nabla\phi = -\frac{1}{2}(A + QA^T Q^{-1})x - \frac{1}{2}(I + QA^T Q^{-1}A^{-1})kf(c^T x).$$

The Fokker-Planck operator associated with the system (1) can be written as

$$L = L_{\text{grad}} + L_{\text{skew}}$$

where

$$L_{\text{grad}}\rho = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}((\nabla\phi)_i\rho) + \frac{1}{2}\sum_{i,j=1}^{n}(BB^T)_{ij}\rho_{x_i x_j} = \frac{1}{2}\sum_{i,j=1}^{n}(BB^T)_{ij}\frac{\partial}{\partial x_i}(e^{-2\phi}\frac{\partial}{\partial x_j}e^{2\phi}\rho)$$

and

$$L_{\text{skew}}\rho = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}[(QA^TQ^{-1} - A)_{ij}x_j\rho] + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}[(QA^TQ^{-1}A^{-1} - I)_{ij}k_j f(c^T x)\rho].$$

Define a *gauge transform* $\widetilde{P}$ of an operator P by $\widetilde{P}\rho = e^{2\phi}P(e^{-2\phi}\rho)$. It is not hard to show directly that

$$\int_{\mathbb{R}^n} \rho \widetilde{L}_{\text{grad}}\rho e^{-2\phi}dx \le 0, \qquad \int_{\mathbb{R}^n} \rho \widetilde{L}_{\text{skew}}\rho e^{-2\phi}dx = 0.$$

Therefore, all eigenvalues of $\widetilde{L}$ with eigenfunctions in the space $\{\rho : e^{-\phi}\rho \in L^2(\mathbb{R}^n)\}$ are nonpositive. This implies that all eigenvalues of L with eigenfunctions in the space $\{\rho : e^{\phi}\rho \in L^2(\mathbb{R}^n)\}$ are nonpositive. Thus it follows that the steady-state density is a stable equilibrium for (20). To obtain specific information about its domain of attraction and the speed of convergence to steady state, one needs to carry out a more detailed spectral analysis of the Fokker-Planck operator L.

Regarding convergence to the steady-state density (17) for the system (12), some results can be found in the literature. Namely, according to Theorem 3 of [8], for $T \to +\infty$ and all $x_0 \in \mathbb{R}^n$ we have

$$\mathcal{P}\left\{\frac{1}{T}\int_0^T g(x(t))dt \to \int_{\mathbb{R}^n} g(x)\rho_{ss}dx \,\middle|\, x(0) = x_0\right\} = 1$$

where ρ_{ss} is the steady-state density and g is any real-valued function integrable with respect to the measure $\rho_{ss}dx$.

6 Non-compatible systems

Compatibility is not a necessary condition for the existence of a steady-state probability distribution. In the previous sections we have discussed systems for which explicit formulae for steady-state probability densities can be found. It would be interesting to try to develop a perturbation theory that would allow us to obtain specific information about steady-state probability distributions for those systems that are not compatible. We discuss some preliminary results here.

Consider the system (1), and denote by L the corresponding Fokker-Planck operator and by L^* its adjoint. Suppose that we have a nonnegative, twice continuously differentiable function $V(x)$ in $\mathbb{R}^n$, which is dominated by a polynomial. Theorem 2 of [8] can now be formulated as follows: *If there exist numbers $R < \infty$ and $k > 0$ such that $L^*V(x) \le -k$ for all x satisfying $\|x\| > R$, then the process defined by (1) admits a steady-state probability distribution.* We can apply this Liapunov-like criterion to establish the existence of steady-state probability distributions at the expense of having to abandon constructive proofs and explicit formulae.

As is well known, if the assumption a) of the Introduction holds, then there exists a positive quadratic function $V(x) = x^T Cx$ whose derivative along the solutions of $\dot{x} = Ax$ is $-x^T Dx$, with D symmetric positive definite. For the system (1) this implies that outside the ball of radius R

$$\begin{aligned} L^*V(x) &= -x^T Dx + 2x^T Ckf(c^T x) + \mathrm{tr}(CBB^T) \\ &\le -\lambda_{min}(D)\|x\|^2 + \mathrm{tr}(CBB^T) \\ &\quad + \max(0, \beta\lambda_{max}(Ckc^T + ck^T C^T)\|x\|^2) \end{aligned}$$

providing that

$$\alpha x^2 \le xf(x) \le \beta x^2 \quad \text{for } \|x\| > R \text{ and some } \beta > \alpha > 0. \tag{21}$$

Here λ_{min} and λ_{max} stand for the minimal and the maximal eigenvalue, respectively. Now we see that for β small enough there exists a steady-state probability distribution.

EXAMPLE 3. Consider the (non-compatible) second-order system

$$\ddot{x} + f(\dot{x}) + x = \dot{w}. \tag{22}$$

Assuming that (21) holds, we can recast (22) as $\ddot{x} + \epsilon\dot{x} + g(\dot{x}) + x = \dot{w}$ with $0 < \epsilon < \alpha$. In the above notation, take $C = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$, where $a > 0$. This gives $Ckc^T + ck^T C^T = \begin{pmatrix} 0 & -1 \\ -1 & -a \end{pmatrix}$ with $\lambda_{max}(Ckc^T + ck^T C^T) = \sqrt{a^2+1} - a$, and $D = \begin{pmatrix} 2 & \epsilon \\ \epsilon & 2(a\epsilon - 1) \end{pmatrix}$. Now, let $\epsilon \to 0$ and $a \to \infty$ in such a way as to have $a\epsilon \to \infty$. Then obviously $\lambda_{min}(D) \to 2$ and $\lambda_{max}(Ckc^T + ck^T C^T) \to 0$, which proves the existence of a steady-state probability distribution for all $\beta > \alpha$.

We now single out a class of nonlinear feedback systems perturbed by white noise for which explicit bounds can be obtained for certain second moments in steady state. These are single-input, single-output systems of the form

$$\dot{x} = \Omega x - bf(b^T x) + b\dot{w} \tag{23}$$

where $\Omega = -\Omega^T$. Let us assume that the condition (21) holds. In this case it may be interpreted as saying that the temperature of the system (23) is between $\frac{1}{2\beta}$ and $\frac{1}{2\alpha}$. Rewriting (23) as

$$\dot{x} = (\Omega - bb^T)x - bg(b^T x) + b\dot{w}$$

one easily verifies that it is not compatible unless $\Omega b = 0$. However, notice that

$$\frac{d}{dt}\mathcal{E}x^T x = -2\mathcal{E}b^T xf(b^T x) + b^T b.$$

Assume for simplicity that $\|b\| = 1$. Providing that the steady-state probability distribution exists, we deduce that in steady state $\mathcal{E}b^T xf(b^T x) = 1/2$ and therefore

$$\frac{1}{2\beta} \le \mathcal{E}(b^T x)^2 \le \frac{1}{2\alpha}.$$

The equation (22) may serve as a simple example.

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