

Quantized feedback systems perturbed by white noise

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Abstract

This paper treats a class of nonlinear feedback systems perturbed by white noise, the nonlinearity being given by a piecewise constant function of a certain type. We obtain explicit formulae for steady-state probability densities associated with such systems. This result is used to address a stochastic optimal control problem that can be interpreted as minimization of the cost of implementing a feedback control law.

1 Introduction

Given an integer $M > 0$ and a real number $\Delta \geq 0$, we define the *quantizer* $q : \mathbb{R} \rightarrow \mathbb{Z}$ by the formula

$$q(x) = \begin{cases} M, & x \geq (M + 1/2)\Delta \\ -M, & x < -(M + 1/2)\Delta \\ \lfloor \frac{x}{\Delta} + \frac{1}{2} \rfloor, & -(M + 1/2)\Delta \leq x < (M + 1/2)\Delta \end{cases}$$

Thus on each interval $J_k := [(k - 1/2)\Delta, (k + 1/2)\Delta)$, where $k \in \mathbb{Z}$ and $-M \leq k \leq M$, the function q takes on the value k .

We consider the quantized feedback system

$$dx = Axdt + Gdw + bq(c^T x)dt \quad (1)$$

where $x, b, c \in \mathbb{R}^n$, w is an m -dimensional Wiener process, and A and G are matrices of appropriate dimensions. All stochastic differential equations in this paper are to be interpreted in the Itô sense. Let us make the following two assumptions regarding the system (1).

- a) All eigenvalues of A have negative real parts.
- b) (A, G) is a controllable pair.

We will denote by L the Fokker-Planck operator associated with (1). The linear case ($b = 0$) is well understood: the steady-state probability density (s.s.p.d.), which is the solution of the equation

$$L\rho(x) = 0 \quad (2)$$

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is a Gaussian with mean 0 and variance Q satisfying

$$AQ + QA^T + GG^T = 0. \quad (3)$$

The nonlinear case is much more difficult. Lyapunov-like methods have been used to prove the existence of s.s.p.d. (this work can be traced back to at least early 60s—see the references in [4] and [5]). However, these results do not provide specific expressions for s.s.p.d. We will formulate a condition on the parameters of the system (1) which enables us to obtain an explicit formula for a s.s.p.d., and use this result to investigate a stochastic optimal control problem associated with the steady-state performance of (1).

We will not embark on the issue of existence of solutions for stochastic differential equations with discontinuous right-hand side as in (1). The situation when instead of a piecewise constant function q one uses a suitable continuous approximation is covered by the standard theory. In fact, the results of the next section directly generalize to that case. The s.s.p.d. associated with (1) is to be understood as a solution of (2) almost everywhere, and can be obtained in the limit as continuous approximations approach q .

2 Compatible systems

Let us denote by Q the positive definite solution of (3). It is not hard to show that the function

$$\rho(x) = Ne^{-\frac{1}{2}(x+A^{-1}bk)^T Q^{-1}(x+A^{-1}bk) + d_k} \text{ if } c^T x \in J_k$$

with arbitrary constants N and d_k satisfies the equation (2) almost everywhere. This function is piecewise Gaussian. Clearly, if $x \in \mathbb{R}$, we can always determine particular values of d_k so as to make ρ continuous. However, this is not necessarily true in the multidimensional case. We will say that the system (1) is *compatible* if the following *compatibility condition* is satisfied:

$$b = \lambda A Q c \quad \text{for some } \lambda \in \mathbb{R}. \quad (4)$$

This condition is precisely what makes it possible to obtain a continuous s.s.p.d. by choosing appropriate constants as explained above. The following result can be proved by a direct calculation not given here due to space constraints.

Theorem 1 *If the compatibility condition (4) is satisfied, then the process described by the system (1) admits*

a steady-state probability distribution with a continuous piecewise Gaussian density.

As we said before, the above result can be generalized to a larger class of nonlinear feedback systems [2]. In [2] it is also shown that, at least for some initial probability distributions, convergence to steady state can be established, and that for an important class of systems the s.s.p.d. is unique.

3 Stochastic optimal control

Let us assume that (1) is compatible, and that $M = 1$:

$$q(x) = \begin{cases} 1, & x \geq \Delta/2 \\ 0, & -\Delta/2 \leq x < \Delta/2 \\ -1, & x < -\Delta/2 \end{cases}$$

We will consider an optimal control problem with a performance criterion formulated in terms of the s.s.p.d. In view of the remarks at the end of Section 2, solutions to such problems provide information about the behaviour of (1) for large times t .

Let \mathcal{E} denote the expectation with respect to the s.s.p.d. Every time the solution trajectory crosses one of the switching hyperplanes $c^T x = \pm\Delta/2$, we need to communicate to the controller a request to change the control value. This reflects the amount of "attention" needed for implementing a given control law (a similar idea is exploited in [1] in the context of deterministic systems with smooth control functions). One might thus be interested in minimizing the number of such crossings per unit interval of time. Making use of the fact that the s.s.p.d. is an even function of x , let us define the attention cost to be $C = 2\mathcal{E}C_{\Delta/2}(c^T x)$, where $\mathcal{E}C_u(\xi)$ stands for the mean number of crossings of a level u per unit time by a scalar stochastic process $\xi(t)$.

Since the expectation is computed with respect to the s.s.p.d., we may treat the process $c^T x(t)$ as stationary, assuming that it "has reached steady state". Therefore, we may use the celebrated Rice's formula for the mean number of crossings [3]

$$\mathcal{E}C_u(\xi) = \frac{1}{\pi} \sqrt{\frac{-r''(0)}{r(0)}} \cdot e^{-u^2/2r(0)}$$

where $r(\tau) = \mathcal{E}\xi(t)\xi(t+\tau)$ is the autocorrelation function associated with a stationary stochastic process $\xi(t)$. In our case $r(\tau) = \lim_{t \rightarrow +\infty} c^T \mathcal{E}x(t)x^T(t+\tau)c$. Let us first study the following question: when is $\mathcal{E}C_{\Delta/2}(c^T x)$ finite? We will need the following easy

Lemma 2 Assume that $r''(\tau)$ exists in some neighborhood of zero, possibly excluding zero itself. Then $\mathcal{E}C_{\Delta/2}(c^T x) < \infty$ if and only if $r'(0) = 0$.

EXAMPLE 1. For the equation

$$dx = -xdt + dw - bq(x)dt, \quad x \in \mathbb{R}, b > 0 \quad (5)$$

we have $r'(0) = -\mathcal{E}x(t)(-x(t) - bq(x(t))) < 0$, hence the condition of Lemma 2 is not satisfied. Thus we see that it is in fact a nontrivial task to construct a control system with a finite attention cost.

Consider a general linear stochastic system

$$\begin{aligned} dx &= Axdt + Gdw \\ y &= c^T x \end{aligned}$$

We have $\lim_{t \rightarrow \infty} \mathcal{E}y(t)y(t+\tau) = \lim_{t \rightarrow \infty} c^T \mathcal{E}x(t)x(t+\tau)c = c^T Q e^{A^T \tau} c$, where Q is the steady-state variance matrix satisfying the equation (3). Therefore $r'(0) = c^T Q A^T c = \langle c, A Q c \rangle$. Premultiplying (3) by c^T and postmultiplying by c , we obtain $2\langle c, A Q c \rangle = -\langle G^T c, G^T c \rangle$. Thus the condition of Lemma 2 is satisfied if and only if $G^T c = 0$. This means that we will have a finite attention cost if all the directions in which the noise can propagate are parallel to the switching hyperplanes. On the other hand, if G is a nonsingular $n \times n$ matrix, the attention cost will always be infinite.

The above discussion suggests replacing (5) by

$$\begin{aligned} dx &= -xdt + dw - bq(y)dt, \\ dy &= \beta xdt - \beta ydt \end{aligned} \quad (6)$$

with $\beta > 0$. This system is compatible, and the corresponding attention cost is finite. It is not very difficult to compute it directly using our knowledge of the s.s.p.d. $\rho(x, y)$ associated with (6). We have

$$\mathcal{E}y^2 = b^2 \mathcal{P}(|y| > \Delta/2) + \frac{\beta}{\beta+1} \left(\frac{1}{2} - \mu b e^{-(\beta+1)\Delta^2/4\beta} \sqrt{\frac{\pi}{\beta+1}} \right)$$

where $\mu = \rho(0, 0)$. The attention cost is given by

$$C = \frac{\sqrt{\beta}}{\pi} \sqrt{\frac{\beta}{2(\beta+1)}} \cdot 2 e^{-\Delta^2/8\mathcal{E}y^2}.$$

We summarize as follows.

Proposition 3 If (4) holds and $r'(0) = 0$, then the attention cost is well defined and finite.

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