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Explicitly Solvable Control Problems With Nonholonomic Constraints

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Abstract

In this paper we describe a family of explicitly solvable optimal control problems involving nonholonomic systems. It appears that the optimal trajectories for these problems need only be piecewise smooth with the smooth arcs being given in terms of a product of matrix exponentials. The solvability of a matrix equations of the form $e^{\Omega}e^{H-\Omega} = X$ plays a key role.

1 Introduction

The model problem discussed by Gaveau [1] and the author [2] has proved to be a useful source of insight about nonholonomic control. In this paper we identify a class of explicitly solvable problems of this type, a class that includes the problem of [1]-[2] as a limiting form. The optimization problem we solve involves systems that evolve on matrix Lie groups. The hypothesis involves the existence of a $\mathbb{Z}_2$ grading on the Lie algebra generated by the control vector fields. The relationship between the space of the control action and the Lie algebra it generates is formulated in theorem 1. A number of examples will be given and in some cases these will be given a geometric interpretation involving holonomy.

Let $\mathcal{G}$ be a real Lie group and let $\mathcal{L}$ be its Lie algebra. We will say that a sum decomposition of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ defines a $\mathbb{Z}_2$ grading if

$$[\mathcal{L}_0, \mathcal{L}_0] \subset \mathcal{L}_0$$

$$[\mathcal{L}_0, \mathcal{L}_1] \subset \mathcal{L}_1$$

$$[\mathcal{L}_1, \mathcal{L}_0] \subset \mathcal{L}_1$$

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We will also use the standard notation

$$ad_L(\cdot) = [L, \cdot]$$

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and use $\langle X, Y\rangle = tr(X^TY)$

Theorem 1: Let $\mathcal{G}$ be a real Lie group and let $\mathcal{L}$ be its Lie algebra. Assume that vector space decomposition $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ defines a $\mathbb{Z}_2$ grading on $\mathcal{L}$. Consider the control system

$$\dot{X}(t) = U(t)X(t)\ ;\ X(0) = I\ ;\ U(t) \in \mathcal{L}_1$$

together with the performance functional

$$\eta = \frac{1}{2}\int_0^1 |\langle U(t), U(t)\rangle| dt$$

and the constraint $X(1) = X_1$. Assuming that the set

$$S = \{(L_0, L_1) | L_0 \in \mathcal{L}_0\ ;\ L_1 \in \mathcal{L}_1\ ;\ e^{L_0}e^{L_1 - L_0} = X_1\}$$

is nonempty, for each $(L_0, L_1) \in S$ the control

$$U(t) = e^{-L_0t}L_1e^{L_0t}$$

renders η stationary. The corresponding value of η is given by

$$\eta = \langle L_1, L_1\rangle$$

and the stationary trajectory is

$$X(t) = e^{L_0t}e^{(L_1 - L_0)t}X(0)$$

The proof is based on the following elementary but useful lemma.

Lemma 1: Let $\mathcal{G}$ be a real Lie group and let $\mathcal{L}$ be its Lie algebra. Suppose that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ defines a $\mathbb{Z}_2$ grading of $\mathcal{L}$. If $\pi_0 : \mathcal{L} \to \mathcal{L}_0$ and $\pi_1 : \mathcal{L} \to \mathcal{L}_1$ is the natural projection, then the equation

$$\dot{M}(t) = [\pi_0(M(t)), M(t)] = -[\pi_1(M(t)), M(t)]$$

has the solution

$$M(t) = e^{\pi_0(M(0)t}M(0)e^{-\pi_0(M(0)t}$$

Proof of Lemma: Observe that the equation can also be written as

$$\dot{M}(t) = [\pi_0(M(t)), \pi_1(M(t))]$$

Appealing to the assumption about the graded structure, we see that the projection of the right-hand side of this equation onto $\mathcal{L}_0$ is zero. Thus the projection of $M(t)$ onto $\mathcal{L}_0$ is constant and $\dot{M}(t) = [\pi_0(M(t)), \pi_1(M(t))]$ can be regarded as a linear equation for $\pi_1(M(t))$. Solving it we get the desired result.

Proof of Theorem: We get the first order necessary conditions by applying the maximum principle of Pontryagin. We represent linear functionals on X as $\phi_P(X)$ =tr(PX) with P being a square matrix. The hamiltonian is then

$$h(X, P, U) = trPUX) + \frac{1}{2}tr(U^T(t)U(t))$$

Thus P satisfies the equation

$$\dot{P}(t) = -P(t)U(t)$$

and

$$U(t) = \pi_1(X(t)P(t))$$

Now introduce $M = XP$. The differential equation for M is

$$\dot{M}(t) = [\pi_1(M(t), M(t)]$$

Applying the lemma we see that

$$\dot{M}(t) = e^{-\pi_0(M(0))t}M(0)e^{-\pi_0(M(0))t}X(t)$$

Consider the variable Z defined as $Z(t) = e^{-\pi_1(M(0))t}X(t)$. The definition implies that Z satisfies a constant coefficient equation

$$\dot{Z}(t) = M(0)Z(t)$$

Solving this, and then substituting back for X, gives the result as clamed.

Remark 1: Of course controllability is a necessary condition to be able to solve $e^{L_0}e^{L_1-L_0} = X_1$. However, there is no assurance that it is sufficient.

2 Gradings on Traceless Matrices

An especially interesting application of this theorem involves the group of matrices having determinant 1, $\mathbb{S}l(n)$, and its Lie algebra $sl(n)$, the set of square matrices whose trace vanishes. Consider splitting $sl(n)$ into the space of skew-symmetric matrices $\mathcal{L}_0$ and the space of traceless symmetric matrices $\mathcal{L}_1$. The subset of $\mathbb{S}l(n)$ that can be expressed as $X = e^{\Omega}e^{H-\Omega}$ with $\Omega = -\Omega^T$ and $H = H^T$ has nonempty interior in $\mathbb{S}l(n)$. This does not follow from basic controllability considerations but will follow from arguments to be given below. In the meantime, observe that if X can be so expressed and if Θ is orthogonal, then

$$\Theta^T X \Theta = e^{\Theta^T \Omega \Theta}e^{\Theta^T(H-\Omega)\Theta}$$

is also expressible in this way. Also note that any symmetric positive definite can be expressed as $e^{\Omega}e^{H-\Omega}$ simply by letting Ω be zero.

Remark 2: If we express an orthogonal matrix as

$$\Theta = e^{\Omega}e^{H-\Omega}$$

with Ω real and skew-symmetric then the eigenvalues of Θ can only be plus or minus one; i.e. $\Theta^2 istheidentity$. To see this notice that if $e^{\Omega}e^{H-\Omega}$ is orthogonal then $e^{H-\Omega}$ must be orthogonal. But the inverse transposse of $e^{H-\Omega}$ is $e^{-H-\Omega}$ and so we require $e^{-H-\Omega} = e^{H-\Omega}$. Suppose that $H - \Omega x_i = \lambda_i x_i$. Then $e^{H-\Omega}x_i = e^{\lambda_i}x_i$ and $e^{-H-\Omega}x_i = e^{\lambda_i}x_i$ as well. Now if e^{λ_i} is unrepeated as an eigenvalue of $e^{H-\Omega}$ then we can assert that x_i is an eigenvalue of $-H-\Omega$ and that its eigenvalue is $\lambda_i + 2\pi i n$ for some integer n. However, subtracting $(-H-\Omega)x_i = \lambda_i = 2\pi i n$ from $(H - \omega)x_i = \lambda_i x_i$ we see that x_i is a purely imaginary eigenvalue of H. Thus it must be zero and H must vanish on any simple eigenvalue of $e^{H-\Omega}$. Orthogonal matrices are orthogonally similar to a real block form with the blocks being two by two. Applying this argument one block at a time we see that either the restriction of H to this sub space vanishes or else the eigenvalues are repeated. If they are repeated then they are either one or minus one. If they are one the matrix is the identity .

Lemma: The expansion of $e^{\epsilon\Omega}e^{\epsilon H-\epsilon\Omega}$ about $\epsilon = 0$ is of the form

$$e^{\epsilon\Omega}e^{\epsilon H-\epsilon\Omega} = I + \epsilon H + \epsilon^2 J + R(\epsilon)$$

with J symmetric and $R(\epsilon)$ of third order.

Proof: Simply expand the two exponentials keeping terms up to and including those of third order. This gives

$$e^{\epsilon\Omega}e^{\epsilon H-\epsilon\Omega} = (I+\Omega+\frac{1}{2}\Omega^2+...)(I+H-\Omega+\frac{1}{2}(H-\Omega)^2+...)$$

Some simplification shows that the second order term is one half $H^2 + [\Omega, H]$ and is, therefore, symmetric.

The significance of this lemma is that it shows that the smooth stationary solutions defined by Theorem one are symmetric not just to first order but also when second order in Ω and H. On the other hand, the control system

$$\dot{X} = UX$$

Is not only controllable on $\mathbb{S}l(n)$ with U tracessess and symmetric, but it is even first bracket controllable in the sense that U and $[U, U']$ span the tangent space. In this situation we know that any point $e^A X_1$ in the neighborhood of X_1 can be reached with a value of η that is at worst linear in $||A||$. Thus we see that trajectories that are symmetric up to third order can not be optimal for steering from I to points of the form e^{Ω}

with Ω small and skew-symmetric. The optimal trajectories for such transfers will necessarily involve large values of Ω.

Example 1: Consider the problems m of steering the system

$$\frac{d}{dt}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} u & v \\ v & -u \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

from $X(0)I$ to $X(1) = -I$ while minimizing the integral of the norm of the control matrix

$$\eta = \int_0^1 2u^2 + 2v^2 dt$$

As a first step we look at the possible solutions of the equation

$$e^{\Omega}e^{H-\Omega} = -I$$

Specializing remark two to the present situation, we see that the only two by two orthogonal matrices that can be expressed as $e^{H-\Omega}$ with H symmetric and Ω skew-symmetric atre $\pm I$ and that the eigenvalues of $H - \Omega$ must be an integer multiple of π. Taking this into account, we see that the real solutions of

$$\exp\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}\begin{bmatrix} a & b-\theta \\ b+\theta & -a \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

occur when $\theta^2 - a^2 - b^2 = (n\pi)^2$. This, coupled with the constraints $\theta = m\pi$ and $m + n$ odd, mean that

$$a^2 + b^2 = (m\pi)^2 - (n\pi)^2$$

The value of $a^2 + b^2$ that minimizes this is

$$a^2 + b^2 = 4\pi^2 - \pi^2 = 3\pi^2$$

Thus the minimum value of η is $6\pi^2$. In this case the optimal control is not unique because only the value of $a^2 + b^2$ is specified, not a and b individually.

Example 2: Consider again the same system but now suppose the goal is to steer the system to an arbitrary orthogonal matrix. We need to solve the equation

$$\exp\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}\begin{bmatrix} a & b-\theta \\ b+\theta & -a \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$

Now we need $\theta = \psi + \pi$, as well as

$$(\pi + \psi)^2 - a^2 - b^2 = \pi^2$$

The value of $a^2 + b^2$ is

$$a^2 + b^2 = (\psi + \pi)^2 - \pi^2 = 2\pi\psi + \psi^2$$

Again, the optimal control is not unique.

Example 3: Consider now the case

$$\dot{X} = UX$$

with the task of steering $X(0) = I$ to a symmetric positive definite matrix Q with det $Q = 1$. In this case we can let $\Omega = 0$ and the minimum cost is the sum of the squares of the lograthims of the eigenvalues of Q

$$\eta = \sum(\log\lambda_i)^2$$

3 A Nilpotent Approximation

We can gain some understanding of the local behavior of this system near $X = I$ by splitting X into its symmetric and skew-symmetric parts and making a suitable approximation. Introduce F, G,

$$F = \frac{1}{2}(X + X^T) - I; G = \frac{1}{2}(X - X^T)$$

whose differential equations are

$$\dot{F}(t) = U(t) + \frac{1}{2}[U(t), G(t)] + \frac{1}{2}(U(t)F(t) + F(t)U(t))$$

$$\dot{G}(t) = \frac{1}{2}[U(t), F(t)] + \frac{1}{2}(U(t)G(t) + G(t)U(t))$$

These are fully coupled, however they have a nilpotent approximation near the point where F and G vanish. An expansion about $F = 0$ and $G = 0$, yields a the system

$$\dot{F}(t) = U(t)$$

$$\dot{G}(t) = \frac{1}{2}[U(t), F(t)]$$

This pair can be thought of as a sub-system of the basic model problem

$$\dot{x}(t) = u(t)$$

$$\dot{z}(t) = x(t)u^T(t) - u(t)x^T(t)$$

and can be understood using known methods. In fact, borrowing from the results of [2], we see that the corresponding variational equation, written in terms of a skew-symmetric matrix of Lagrange multipliers, Ω is

$$\ddot{F}(t) + [\Omega, \dot{F}(t)] = 0$$

Solving this yeilds

$$\dot{F}(t) = e^{\Omega t}\dot{F}(0)e^{-\Omega t}$$

and

$$F(t) = e^{\Omega t}Ae^{-\Omega t} + B$$

with $F(0) = A - B$ and $[\Omega, A] = \dot{F}(0)$. The case $F(0) = 0$ is especially interesting. In this case we have

$$G(t) = \frac{1}{2}\int_0^t [e^{\Omega t}[\Omega, A]e^{-\Omega t}, e^{\Omega t}Ae^{-\Omega t} - A]dt$$

Remark: The minimum cost required to steer the system

$$\dot{F}(t) = U(t)$$

$$\dot{G}(t) = \frac{1}{2}[U(t), F(t)] \; ; \; U = U^T$$

from $(F(0), G(0)) = (0, 0)$ to $(F(1), G(1)) = (0, S)$ is 4π times the sum of the magnitudes eigenvalues of S. To prove this it is enough to observe that If F and G are two by two then the equations of motion are

$$\dot{f}_{11} = u_{11}$$

$$\dot{f}_{12} = u_{12}$$

$$\dot{g}_{12} = u_{11}f_{12} - u_{12}f_{11}$$

and for this system the result is known. For the general system, observe that we can make use of the orthogonal invariance of the cost functional to assume that S is in real block diagonal form and that in this case the cost associated with each subsystme simply adds. This is in strong contrast with the behavior of the complete distance-area system, $\dot{x} = u$; $\dot{Z} = xu^T - ux^T$ whose performance is more complicated. (See [2].)

Remark 3: In terms of the variables $Z = XX^T$ and $\Theta = X\sqrt{Z}^{-1}$ the control system $\dot{X} = UX$ with U symmetric takes the form

$$\dot{Z}(t) = U(t)Z(t) + Z(t)U(t)$$

$$\dot{\Theta} = -\sqrt{Z}^{-1} ad_{\sqrt{Z}}^{-1}(UZ + ZU)\Theta + \sqrt{Z}^{-1}U\sqrt{Z}\Theta$$

Theorem one then asserts that $Z(t) = e^{At}(e^{At})^T$ and that Θ is constant if A is symmetric.

4 Gradings on Skew-symmetric Matrices

Our Second example concerns the group of n by n orthogonal matrices having determinant +1. We denote this group by $\mathbb{S}o(n)$. The corresponding Lie algebra is the set of n by n skew-symmetric matrices, $so(n)$. We work with the quadratic form $\langle\Omega, \Omega\rangle = \text{tr}\Omega^T\Omega$. Of course $\mathbb{S}o(n)$ contains $\mathbb{S}o(n-1)$ as a subgroup and $so(n)$ contains $so(n-1)$ as a Lie sub algebra. We grade $so(n)$ by identifying $\mathcal{L}_0$ with the skew-symmetric matrices whose first column and first row is zero but is otherwise arbitrary. Take $\mathcal{L}_1$ to be the set of n by n skew-symmetric matrices that are zero except for the first row and first column. A short calculation shows that this splitting of the space of skew-symmetric matrices defines a $\mathbb{Z}_2$ grading.

Example 4: Given $X_1 \in \mathbb{S}o(n)$ and given

$$\dot{X}(t) = U(t)X(t) \ ; \ X(0) = I \ ; \ U(t) \in \mathcal{L}_1$$

There exists a control $U(\cdot)$ on $[0, 1]$ that renders stationary the functional

$$\eta = \int_0^1 trU^T(t)U(t)dt$$

subject to $X(1) = X_1$ provided we can solve

$$e^{\Omega}e^{S-\Omega} = X_1 \ ; \ \Omega \in \mathcal{L}_0 \ ; \ S \in \mathcal{L}_1$$

This control generates trajectories of the form

$$X(t) = e^{\Omega t}e^{(S-\Omega)t}X(0)$$

Remark 4: In this case the expansion of the product takes the form

$$e^{\Omega t}e^{(S-\Omega)t} = (I+\Omega+\frac{1}{2}\Omega^2+...)(I+S-\Omega+\frac{1}{2}(S-\Omega)^2+...)$$

which expands to

$$e^{\Omega t}e^{(S-\Omega)t} \approx I + S + \frac{1}{2}S^2 + \frac{1}{2}[\Omega, S]$$

Again the bracket term $[S, \Omega]$ points in the direction of $\mathcal{L}_1$.

Remark 5: The system of differential equations can be "block triangularized" in terms of a $n-1$-dimensional part evolving on the unit sphere and a $(n-1)(n-2)/2$-dimensional part evolving on evolving on the space of orthogonal matrices $\mathbb{S}O(n-1)$. Consider the division of X into sub blocks

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & X_{22} \end{bmatrix}$$

Let x_1 denote the first column of X, $x = Xe_1$. Then we see that the first column of X satisfies the equation

$$\dot{x}_1(t) = U(t)x_1(t)$$

and can be thought of as evolving on the sphere $\mathbb{S}^{n-1}$. It is then possible to recast the optimization problem in terms of a shortest path problem, with an auxiliary constraint on the holonomy it generates.

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5 References

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