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Stochastic Analysis for Fluid Queueing Systems

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Abstract

The existing literature on network and queuing analysis has a tendency to move as quickly as possible away from the flow equations to a description of the problem in terms of probabilities, leaving behind the sample path description. In this paper we formulate and solve a number of general questions in this area using sample path methods as an important part of the process. Relying, as it does, on the theory of stochastic differential equations, this approach brings to bear a heretofore ignored but quite effective problem solving methodology. It also serves to bring the subject of controlling queues in closer contact with other branches of automatic control.

1 Introduction

Inspired, at least in part, by the successful use of Wiener processes in estimation and stochastic control, over the last thirty years it has been often suggested that methods based on stochastic differential equations should find an important role in queueing theory. Even so, the accumulation of useful technique and solved problems has remained rather modest. Undaunted, we take up this challenge. Presented here is a set of solved problems based on techniques involving stochastic differential equations with Poisson counters and the related hyperbolic partial differential equations which describe the evolution of the probability distributions. From the queueing point of view, our models involve flows with finite buffers, flows with priorities, tandem configurations, multi-class flows, etc. The space available for this paper does not permit a full presentation of both the theory and a fully representative set of worked examples. In hopes of offsetting the image of stochastic control as being stronger on theory than it is on solvable problems, we devote most of the space to examples. We hope to present a more complete account in the near future.

Our central theme is that by passing back and forth between a description of the sample paths using differential equations and a description of the evolution of the probability density in terms of the Fokker-Plank equation, we can both unify and simplify analytical aproachs to these problems. In the treatment given here we use Poisson counter driven stochastic differential equations together with a systematic use of the Itô calculus as the fundamental modeling methodology. The idea of using stochastic differential equations to model queueing delays is not new. Deterministic fluid model has been suggested by several authors including Kleinrock, [7]. Anick et al. [1] seems to be the first to analyze a stochastic fluid queueing system.The relevant underlying theory is covered in Brémaud [3] (and, together with Martins-Neto/Wong [8] among the first) expositions of the dynamic systems view of queueing systems. Martins [8] treats some classical queueing problems with the martingale calculus and Davis [6] treats in detail piecewise deterministic process.

2 Mathematical Preliminaries

2.1 Poisson Driven Differential Equations

Poisson counter is a simple but important process. In [4], the utility of the Poisson Counter is greatly extended by combining them with ideas from differential equations. We now briefly describe this approach to pave the way for our applications.

Consider a stochastic integral equation

$$x(t) = x(0) + \int_0^t f(x(\tau),\tau)d\tau + \int_0^t g(x(\tau),\tau)dN_\tau \quad (1)$$

where N_τ is a Poisson Counter. The solution of equation (1) is defined as follows.

Definition: $x(\cdot)$ is a solution of (1) in the Itô sense if, on an interval where N is constant, x satisfies $\dot{x} = f(x,t)$ and if, when N jumps at t_1, x changes according to

$$\lim_{t\to t_1^+} x(t) = g(\lim_{t\to t_1^-} x(t), t_1) + \lim_{t\to t_1^-} x(t) \quad (2)$$

and $x(\cdot)$ is taken to be continuous from the left. Equation (1) is often written as

$$dx(t) = f(x,t)dt + g(x)dN \quad (3)$$

and is called the Poisson Counter driven stochastic Differential Equation.

Some consequences of the above definition are listed below. In the paper [5] there appears an explicit construction of a sample path realization of an arbitrary finite state, continuous time jump process using differential equations driven by Poisson pulse trains. We will use this idea extensively because it can be combined with ordinary differential equation models leading to a uniform description of interesting situations.

Consider a stochastic differential equation driven by n independent Poisson Counters $N_1, ..., N_n$:

$$dx = f(x)dt + \sum_{i=1}^{n} g_i(x)dN_i, \quad x \in R^n. \tag{4}$$

We have the following Itô rule.

If $\psi : R^n \to R$ is a differentiable function, then

$$d\psi(t) = \langle \frac{\partial \psi}{\partial x}, f(x)\rangle dt + \sum_{i=1}^{n}[\psi(x(t)+g_i(x(t))) - \psi(x(t))]dN_i. \tag{5}$$

This rule will be further explained in some of the concrete application examples later. Since $x(t)$ is continuous from the left and the Poisson counter is taken to be continuous from the right, we have

$$\frac{d}{dt}\mathcal{E}x(t) = \mathcal{E}f(x(t)) + \sum_{i=1}^{m}(\mathcal{E}g_i(x(t),t))a_{ii} \tag{6}$$

where a_{ii} is the rate for N_i.

Furthermore, if $x(t)$ has a smooth density function $\rho(t,x)$, then we would have the following equation similar to the Fokker-Planck equation for the Wiener process driven systems.

$$\frac{\partial \rho(t,x)}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho(t,x)] +$$

$$\sum_{i=1}^{m} a_i \left(\rho(t, \hat{g}_i^{-1}(x)) \left| \det(I + \frac{\partial g_i}{\partial x}) \right|^{-1} - \rho(t,x) \right) \tag{7}$$

where $\hat{g}_i(x) = x + g_i(x)$ and $\hat{g}_i^{-1}(x)$ is the value of x just before the jump of N_i [4]. (If the inverse of $\hat{g}$ has more than one branch then it is necessary to sum over all inverse images of x.)

2.2 Hyperbolic Differential Equations

As just discussed, the evolution equations for the probability density functions of interest in the study of Poisson counter driven differential equations are first order in time and space, usually involving nonlocal effects in that the coefficients of the partial differential operators may take the form $a(x-b)\frac{\partial}{\partial x}$. Such partial differential-difference equations are notoriously intractable. However, and this is what makes it possible to solve many problems involving fluid queues, is that if the coefficients are constant then these nonlocal effects go away. Constancy of the coefficients arises when the flow rates are independent of the queue length, as long as the queue length is positive. In this case the Fokker-Planck equation can be cast as a set of simultaneous first order partial differential equations of the form

$$\frac{\partial}{\partial t}\rho(t,v) = \left(\sum A_i \frac{\partial}{\partial x_i} + A_0\right)\rho(t,x) = 0 \tag{8}$$

with appropriate boundary conditions. Even for this subclass, when more than one dimension is involved the time dependent theory is involved. When conditions for the existence of a steady state are assumed, however, the asymptotic theory can be worked out. These additional assumptions are analogous to the stability conditions associated with diffusion processes. As illustrated by the examples to be given, the conditions for the existence of a steady state are often rather intuitive and of independent significance. Garabedian [11] provides a context in which to think about hyperbolic systems.

3 Hybrid Models

The models we consider here are stochastic differential equations driven by continuous time jump processes taking the form

$$dx = \sum \phi_i(x)dN_i \;\; ; \;\; x(0) \in S \tag{9}$$

$$dv = f(x,v)dt + \sum g_i(x,v)dN_i \tag{10}$$

The N's are Poisson counters. An important aspect of this model is that the assumption that S is a finite set does not make their theory either trivial or uninteresting. It does mean that the functions ϕ_i must be chosen to be compatible with S in the sense that for all x_j in S and for all i, we assume that$\phi_i(x_j) - x_j \in S$. These (x,v) models have a "hybrid" character in that x takes on values in a finite set whereas v takes on values in a continuum. It is known that any finite state jump process can be generated by the x part of such a system; a general construction was given in [5].

In the classical "virtual work" approach to the Poisson arrival, deterministic service rate model, the x equation is absent and the v equation is driven directly by the Poisson counter. We can think of such models as *Pulse Driven Models.* By way of contrast, the fluid queueing models to be treated here have the property that the x equation is present and the g_i are absent. We can think of these as being *Level Driven Models*

Because x evolves independently of v it is possible to evaluate the statistical properties of x independently of any analysis of v. Thus average inflow rates can be determined before considering f. Because the large time asymptotic behavior of the solutions often depend on the capacity of the system modeled by f to handle the inflow, modeled by x, this is important.

4 Examples

Our first example is a familiar one, having been the subject of the paper by Anick, Mitra and Sondhi [1]. Our reason for presenting it is to show that it can also be done with path calculus for moments while some later examples can only be done with moments. It also provides a transition point and a means of introducing the notation and language to be used later. Here and below, we use boldface I to denote indicator functions. The specific notation $\mathbf{I}_v$ is the indicator function for set $v > 0$..

1. The Basic Single Server Infinite Buffer Queue Consider an on-off Markov modulated source fed into a queue with constant service rate c. The model is

$$\begin{cases} dx(t) = (1-x(t))dN_1 - x(t)dN_2 \ ; \ x(0) \in \{0,1\} \\ dv(t) = -c\mathbf{I}_v dt + hx(t)dt \end{cases} \tag{11}$$

with $h > c > 0$. Let the rates of the counters N_1 and N_2 be a_{11} and a_{22}, respectively. If p_0 and p_1 are the respective probabilities associated with $x = 0$ and $x = 1$ then the probability law for x is generated by

$$\begin{bmatrix} \dot{p}_0 \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} -a_{11} & a_{22} \\ a_{11} & -a_{22} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \tag{12}$$

From this we see that the flow rate into the v system has an average value of $ha_{11}/(a_{11}+a_{22})$. Unless this is less than the outflow capacity c the size of v will grow without bound. Because x only takes on the values 0 and 1 it is convenient to abbreviate the joint density $\rho(t,0,v)$ as $\rho_0(t,v)$ and $\rho(t,1,v)$ as $\rho_1(t,y)$. Standard arguments (or the formula in section 2) then lead to the Fokker-Planck equation

$$\begin{bmatrix} \frac{\partial\rho_0(t,v)}{\partial t} \\ \frac{\partial\rho_1(t,v)}{\partial t} \end{bmatrix} = \begin{bmatrix} c\frac{\partial}{\partial v} - a_{11} & a_{22} \\ a_{11} & (c-h)\frac{\partial}{\partial v} - a_{22} \end{bmatrix} \begin{bmatrix} \rho_0(t,v) \\ \rho_1(t,v) \end{bmatrix}. \tag{13}$$

This equation is to be solved on the domain $(t,v) \in ([0,\infty),[0,\infty))$ subject to a boundary condition at $v = 0$ which relates the strength of the impulse at $v = 0$ to $\rho(t,0^+)$. More specifically,

$$a_{11} \lim_{\epsilon\to 0} \int_0^\epsilon \rho_0(t,v)dv = (h-c)\rho(t,0^+) \tag{14}$$

The first step in looking for an equilibrium solution is to find the solutions of the determinental equation

$$\det \begin{bmatrix} c\lambda - a_{11} & a_{22} \\ a_{11} & (c-h)\lambda - a_{22} \end{bmatrix} = 0 \tag{15}$$

One solution is $\lambda_1 = 0$ and the other is

$$\lambda_2 = \frac{(a_{11}+a_{22})c - a_{11}h}{c(c-h)} \tag{16}$$

Note that λ_2 is negative exactly when the the capacity of the buffer exceeds the expected input flow. This suggests that we look for a solution of the form

$$\begin{bmatrix} \rho_{B,0}(v) \\ \rho_{B,1}(v) \end{bmatrix} = \begin{bmatrix} \hat{g}\delta(v) + \hat{f}e^{\lambda_2 v} \\ \hat{k}e^{\lambda_2 v} \end{bmatrix}. \tag{17}$$

The expression for the vector density is

$$\begin{bmatrix} \rho_0(v) \\ \rho_1(v) \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{a_{11}h}{c(a_{11}+a_{22})}\right)\delta(v) + \frac{(h-c)a_{11}}{c(a_{11}+a_{22})}\lambda_2 e^{\lambda_2 v} \\ \frac{a_{11}}{a_{11}+a_{22}}\lambda_2 e^{\lambda_2 v} \end{bmatrix} \tag{18}$$

The marginal density for v is the sum of these components, which, of course, is the same as the results of Anick et al.

In some less elementary situations it still possible to compute the moments of the distribution even though the Fokker-Planck equation can not be solved directly. To facilitate comparison we now rework this model showing how to compute the steady state value or the moments directly from the sample path equations. Observe that the differential equations imply, via the Itô calculus, that

$$\frac{d}{dt}\mathcal{E}v^2 = -2c\mathcal{E}v + 2h\mathcal{E}xv \tag{19}$$

$$\frac{d}{dt}\mathcal{E}vx = \mathcal{E}(1-x)va_{11} - \mathcal{E}xva_{22} - c\mathcal{E}x + h\mathcal{E}\mathbf{I}_v x \tag{20}$$

It is important to observe that $\mathcal{E}x\mathbf{I}_v = \mathcal{E}x$, because v is positive whenever x is. Thus if a steady state exists then the following sum must vanish.

$$\begin{bmatrix} a_{11} & -(a_{11}+a_{22}) \\ -2c & 2h \end{bmatrix} \mathcal{E} \begin{bmatrix} v \\ xv \end{bmatrix} + \mathcal{E} \begin{bmatrix} (h-c)\frac{a_{11}}{a_{11}+a_{22}} \\ 0 \end{bmatrix} \tag{21}$$

Thus $c\mathcal{E}v = h\mathcal{E}xv$ and when the capacity condition $(h-c)a_{11} \le ca_{22}$ holds we have

$$\left(1 - \frac{c(a_{11}+a_{22})}{h}\right)\mathcal{E}v = (c-h)\mathcal{E}x \tag{22}$$

or,

$$\mathcal{E}v = \left(c - \frac{ha_{11}}{a_{11}+a_{22}}\right)^{-1}(h-c)\mathcal{E}[hx]. \tag{23}$$

The nth moments of steady state v can be obtained quite similarly and the result is

$$\mathcal{E}v^n = \left(c - \frac{ha_{11}}{a_{11}+a_{22}}\right)^{-1} \frac{h-c}{a_{11}+a_{22}} cn\mathcal{E}v^{n-1}, \ n > 1. \tag{24}$$

2. Tandem Servers with Infinite Buffers

Now consider the case of a tandem fluid model. Our solution here seems to be new. The sample path description is

$$\begin{cases} dx(t) = & (1-x(t))dN_1 - x(t)dN_2 \\ dv_1(t) = & -c_1\mathbf{I}_{v_1}dt + c_0x(t)dt \\ dv_2(t) = & -c_2\mathbf{I}_{v_2}dt + c_1\mathbf{I}_{v_1}dt \\ dv_3(t) = & -c_3\mathbf{I}_{v_3}dt + c_2\mathbf{I}_{v_2}dt \\ & \vdots \\ dv_n(t) = & -c_n\mathbf{I}_{v_n}dt + c_{n-1}\mathbf{I}_{v_{n-1}}dt \end{cases} \tag{25}$$

where $\mathbf{I}_v$ denotes the relevant indicator function, taking on the value one for $v \geq 0$. To avoid trivialities we assume $c_0 > c_1 > c_2 > \cdots > c_n > 0$. (If a down stream link has higher bandwidth than its source then we may as well neglect this link). Stability, on the other hand, demands that $c_n \geq c_0\mathcal{E}x$.

It is clear that the probabilistic description of v_1 is the same as it was in the case of the single server. Focusing on v_2, consider

$$dv_2^2 = -2c_2v_2\mathbf{I}_{v_2}dt + 2c_1v_2\mathbf{I}_{v_1}dt.$$

In steady state this leads to

$$c_2\mathcal{E}v_2 = c_1\mathcal{E}v_2\mathbf{I}_{v_1} \tag{26}$$

or

$$\mathcal{E}v_2\mathbf{I}_{v_1} = \frac{c_2}{c_1}\mathcal{E}v_2. \tag{27}$$

Now lets consider a differential equation for $v_1(t)v_2(t)$. We have

$$dv_1v_2 = v_1dv_2 + v_2dv_1. \tag{28}$$

Plugging in the expressions for dv_1 and dv_2, taking the expectations and considering in steady state $d\mathcal{E}v_1v_2/dt = 0$ we have

$$-c_1\mathcal{E}v_2\mathbf{I}_{v_1} + h\mathcal{E}v_2x - c_2\mathcal{E}v_1\mathbf{I}_{v_2} + c_1\mathcal{E}v_1\mathbf{I}_{v_1} = 0. \tag{29}$$

Let us analyze the four terms on the left hand side. For the first term we have from the above that

$$-c_1\mathcal{E}v_2\mathbf{I}_{v_1} = -c_2\mathcal{E}v_2. \tag{30}$$

For the third term we have, noticing that $v_2 = 0$ implies $v_1 = 0$, that

$$-c_2\mathcal{E}v_1[1-\mathbf{I}_{(v_2=0)}] = -c_2\mathcal{E}v_1 - c_2\mathcal{E}v_1\mathbf{I}_{v_2=0} = -c_2\mathcal{E}v_1. \tag{31}$$

The fourth term is easier to see:

$$c_1\mathcal{E}v_1\mathbf{I}_{v_1} = c_1\mathcal{E}v_1. \tag{32}$$

The second term involves $\mathcal{E}v_2x$. Lets consider the differential for v_2x. Since v_2 is continuous we have $dv_2x = v_2dx + xdv_2$ which leads to, after taking expectations,

$$\frac{d}{dt}\mathcal{E}v_2x = a_{11}\mathcal{E}v_2 - (a_{11}+a_{22})\mathcal{E}v_2x - c_2\mathcal{E}x\mathbf{I}_{v_2} + c_1\mathcal{E}x\mathbf{I}_{v_1}. \tag{33}$$

Since $x = 1$ implies both $v_1 > 0$ and $v_2 > 0$ the last two terms on the right hand side are $c_2\mathcal{E}x$ and $c_1\mathcal{E}x$, respectively. Thus we see that $\mathcal{E}v_2x$ can be expressed in terms of $\mathcal{E}v_2$. Finally in steady state we have

$$\mathcal{E}v_2x = \frac{a_{11}}{a_{11}+a_{22}}\mathcal{E}v_2 + \frac{c_1-c_2}{a_{11}+a_{22}}\mathcal{E}x. \tag{34}$$

Substituting this into (29) we have a recursive formula for v_2:

$$\left(\frac{ha_{11}}{a_{11}+a_{22}} - c_2\right)\mathcal{E}v_2 + \frac{h(c_1-c_2)}{a_{11}+a_{22}}\mathcal{E}x + (c_1-c_2)\mathcal{E}v_1 = 0. \tag{35}$$

In order to bring out the structure of the general situation we now organize this point of view in terms of the underlying linear algebra. We require a series of vectors,

$$m_1 = x \;;\; m_2 = \begin{bmatrix} xv_1 \\ I_1v_1 \end{bmatrix} \;;\; m_3 = \begin{bmatrix} xv_2 \\ I_1v_2 \\ I_2v_2 \end{bmatrix} \cdots \tag{36}$$

which can be organized into an equation that defines the steady state

$$0 = \begin{bmatrix} B_{11} & 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \mathcal{E} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \cdots \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \cdots \end{bmatrix} \tag{37}$$

The diagonal blocks, B_{ii} have a regular form:

$$B_{11} = -a_{11} \;;\; B_{22} = \begin{bmatrix} -(a_{11}+a_{22}) & a_{11} \\ c_0 & -c_1 \end{bmatrix} \;;$$

$$B_{33} = \begin{bmatrix} -(a_{11}+a_{22}) & 0 & a_{11} \\ c_0 & -c_1 & -c_2 \\ 0 & c_1 & -c_2 \end{bmatrix} \;;\; \ldots \tag{38}$$

In general we have

$$\left(\frac{ha_{11}}{a_{11}+a_{22}} - c_n\right)\mathcal{E}v_n + \frac{h(c_1-c_n)}{a_{11}+a_{22}}\mathcal{E}x \tag{39}$$

$$+(c_{n-1}-c_n)\mathcal{E}(v_{n-1}+\cdots+v_1) = 0. \tag{40}$$

3. Single Server with Multi-rate Sources

Consider a source that has three possible input flow rates. For example, there may be no flow, a low data rate stream or a high data rate stream. See also [1]. The differential equation model takes the form

$$dx = \sum \phi(x)dN_i \;;\; x \in \{x_1, x_2, \cdots, x_m\} \tag{41}$$

$$dv = -cI_v + x(t)dt \tag{42}$$

In the case $n = 3$ the Fokker-Planck equation takes the form of $\dot{\rho} = A\rho$ where A is given by

$$\begin{bmatrix} (c-x_1)\frac{\partial}{\partial v} - a_{11} & a_{12} & a_{13} \\ a_21 & (c-x_2)\frac{\partial}{\partial v} - a_{22} & a_{23} \\ a_{31} & a_{32} & (c-x_3)\frac{\partial}{\partial v} - a_{33} \end{bmatrix} \tag{43}$$

The Fokker-Planck equation is of the general form

$$\frac{\partial\rho(t,x)}{\partial t} = \left(\frac{\partial}{\partial v}C + A\right)\rho(t,x) \tag{44}$$

Suppose we take the v-domain to be $[0,\infty)$. Then the steady state solution takes the form

$$\rho(v) = e^{C^{-1}Av}\rho_0 + \hat{\rho}\delta(v) \tag{45}$$

The pair $(\rho_0,\hat{\rho})$ represent a total of $2n$ undetermined constants. These are fixed by two different types of conditions. In the first place, the x-marginal, must be a steady state probability vector. That is

$$p_\infty = \int_0^\infty e^{C^{-1}Av}\rho_0 + \hat{\rho}\delta(v)dv \tag{46}$$

should satisfy

$$Ap_\infty = 0. \tag{47}$$

Assuming that A is irreducible, this together with normalization represents n conditions. The other n conditions come from an analysis of the flow of possibility at the junction at $v = 0$. In equilibrium, the net flow rate of probability into each of the n sets $S_i = \{(x,v)|x = x_i, v = 0\}$ must be matched by the flow rate out of the set. Flow out comes about when v is zero and the state of x changes in such a way as to force $v(t)$ to increase. This rate is $A\hat{\rho}$. The flow rate into the S_i is zero if the i^{th} component of $C\rho(0^+)$ is positive but otherwise it is the i^{th} component of $C\rho(0^+)$. If z is a n-dimensional vector we let $[z]_-$ denote the n dimensional vector whose components are either zero or equal to the components of z, depending on whether or not the components of z are positive or negative. Adopting this notation, we may say

$$C\rho(0^+)]_- = A\hat{\rho} \tag{48}$$

This equation represents n more conditions on the pair $(\rho_0,\hat{\rho})$. Finite buffer case can be treated similarly. See also [10].

4. Single Server with Priority Services

Priority, or differentiated service, is becoming reality in Internet. We now consider a deterministic server with capacity c fed by two independent Markov on-off processes x_1 and x_2. Suppose x_1 has absolute priority over x_2. We have:

$$\begin{cases} dx_1(t) = & (1-x_1(t))dN_{11} - x_1(t)dN_{12} \\ dx_2(t) = & (1-x_2(t))dN_{21} - x_2(t)dN_{22} \\ dv_1(t) = & -c\mathbf{I}_{v_1}dt + h_1x_1(t)dt \\ dv_2(t) = & -c\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1})dt + h_2x_2dt. \end{cases} \tag{49}$$

To make the problem meaningful we require that $c < \min(h_1,h_2)$. It is clear that v_1 is the same as before. We focus on $\mathcal{E}v_2$. Consider the following equations.

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{E}v_2^2 = 2c[-\mathcal{E}v_2\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) + h_2\mathcal{E}v_2x_2] \\ &= -2c\mathcal{E}v_2 + 2c\mathcal{E}v_2\mathbf{I}_{v_1} + h_2\mathcal{E}v_2x_2\end{aligned} \tag{50}$$

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{E}v_2v_1 = \mathcal{E}(v_2dv_1 + v_1dv_2) \\ &= -c\mathcal{E}v_2\mathbf{I}_{v_1} + h_1v_2x_1 - c\mathcal{E}v_1\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) + h_2\mathcal{E}v_1x_2 \\ &= -c\mathcal{E}v_2\mathbf{I}_{v_1} + h_1\mathcal{E}v_2x_1 + h_2\mathcal{E}v_1x_2.\end{aligned} \tag{51}$$

since $\mathcal{E}v_1\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) = 0$.

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{E}v_2x_1 = \mathcal{E}(v_2dx_1 + x_1dv_2) \\ &= a_{11}\mathcal{E}v_2 - a_{11}\mathcal{E}v_2x_1 - a_{12}\mathcal{E}v_2x_1 - c\mathcal{E}x_1\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) + h\mathcal{E}x_1x_2 \\ &= a_{11}\mathcal{E}v_2 - (a_{11}+a_{12})\mathcal{E}v_2x_1 + h_2\mathcal{E}x_1x_2\end{aligned} \tag{52}$$

Here $\mathcal{E}x_1\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) = 0$ is because $v_1 = 0$ implies $x_1 = 0$.

$$\begin{aligned}&\tfrac{d}{dt}\mathcal{E}v_2x_2 = \mathcal{E}(x_2dv_2 + v_2dx_2) \\ &= -c\mathcal{E}x_2\mathbf{I}_{v_2}(1-\mathbf{I}_{v_1}) + h_2\mathcal{E}x_2^2 + a_{21}\mathcal{E}v_2(1-x_2) - a_{22}\mathcal{E}v_2x_2 \\ &= -c\mathcal{E}x_2\mathbf{I}_{v_2} + c\mathcal{E}x_2\mathbf{I}_{v_2}\mathbf{I}_{v_1} + h_2\mathcal{E}x_2 + a_{21}\mathcal{E}v_2 - (a_{21}+a_{22})\mathcal{E}v_2x_2 \\ &= -c\mathcal{E}x_2 + c\mathcal{E}x_2\mathbf{I}_{v_1} + h_2\mathcal{E}x_2 + a_{21}\mathcal{E}v_2 - (a_{21}+a_{22})\mathcal{E}v_2x_2.\end{aligned} \tag{53}$$

Note that here we have $\mathcal{E}x_2\mathbf{I}_{v_2} = \mathcal{E}x_2$ and $\mathcal{E}x_2\mathbf{I}_{v_2}\mathbf{I}_{v_1} = \mathcal{E}x_2\mathbf{I}_{v_1}$ because v_2 is positive whenever x_2 is.

Assuming that the steady state exits leads to 4 linear equations. Note that $\mathcal{E}v_1x_2 = \mathcal{E}v_1\mathcal{E}x_2$ and that $\mathcal{E}x_1x_2 = \mathcal{E}x_1\mathcal{E}x_2$ due to apparent independence. There are now 4 unknowns involved in the 4 equations above. They are

$$\mathcal{E}v_2, \mathcal{E}v_2x_1, \mathcal{E}v_2x_2, \mathcal{E}v_2\mathbf{I}_{v_1}. \tag{54}$$

To see that the above equations are independent we list them in a matrix form.

$$\begin{bmatrix} -2c & 0 & h_2 & 2c \\ 0 & h_1 & 0 & -c \\ a_{11} & -(a_{11}+a_{12}) & 0 & 0 \\ a_{21} & 0 & -(a_{21}+a_{22}) & 0 \end{bmatrix} \begin{bmatrix} \mathcal{E}v_2 \\ \mathcal{E}v_2x_1 \\ \mathcal{E}v_2x_2 \\ \mathcal{E}v_2\mathbf{I}_{v_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -h_2\mathcal{E}v_1x_2 \\ -h_2\mathcal{E}x_1x_2 \\ (c-h_2)\mathcal{E}x_2 - c\mathcal{E}x_2\mathbf{I}_{v_1} \end{bmatrix} \tag{55}$$

It is clear that in general the coefficient matrix is non-singular.

5. Service Rate Dependent on Backlog

In the above fluid queueing systems the service rates are constant. The path calculus approach is capable of handling more complex service schemes. For example consider a Markov on-off process feeding into a work-conservative queue with the service rate proportional to the instant workload, $v(t)$. This bandwidth allocation scheme could be useful for certain guaranteed quality service or can be used to approximate a queue with many different priorities. In wireless network this could model a power control scheme. Note that here the queue content is always finite although there is no hard buffer limit. The sample path of this fluid queue is:

$$\begin{cases} dx = & (-x+1)dN_1 - xdN_2. \\ dv(t) = & -cv(t)\mathbf{I}_v dt + xdt. \end{cases} \tag{56}$$

where c is a constant.

Here we use the path calculus to compute the moments. We have

$$\mathcal{E}v = \frac{1}{c}\frac{a_{11}}{a_{11}+a_{22}}, \quad (57)$$

$$\mathcal{E}v^{n+1} = \frac{cn+a_{11}}{c^2n+a_{11}c+a_{22}c}\mathcal{E}v^n, n>0. \quad (58)$$

6. The TCP Window Process

Now consider the application of sample path calculus to TCP traffic. In Misra et al. [9] evidence is given lending support to the idea that most loss traces are close to Poisson streams. There are two different kinds of losses, triple duplicate ack (TD) losses and time out losses (TO). The window size goes on increasing linearly with every RTT till the time loss occurs. For a TD kind of loss the window size is reduced to half it's current value whereas for a TO loss the window size is reduced to 1.

Let $W(t)$ be the window size. Then,

$$dW(t) = \frac{dt}{RTT} + (-W(t)/2)dN_{TD} + (1-W(t))dN_{TO}. \quad (59)$$

To take into account max window size, (59) is modified by multiplying the first term by an indicator function, i.e.,

$$dW(t) = \mathbf{I}_M(W(t))\frac{dt}{RTT} + (-W(t)/2)dN_{TD} + (1-W(t))dN_{TO} \quad (60)$$

where

$$I_M(W(t)) = \begin{cases} 1, W(t) < M; \\ 0, W(t) = M. \end{cases}$$

This ensures that the window size doesn't grow once it has reached M.

The differential equation approach leads to a closed form formula that fits the empirical data extremely well. To simplify the notion we use a_1 and a_2 to denote the rates for N_{TD} and N_{TO}, respectively. The formula is as follows.

$$\mathcal{E}W = \{\frac{1}{RTT}(1-P_m)+a_1\}/(a_2/2+a_1) \quad (61)$$

where, with $K = 1/RTT$ and TO denoting the time out period,

$$P_m = \frac{2a_1^2+2a_1+a_1a_2+2a_1K+2K^2+2K}{(K+1)(2Ma_1+Ma_2+2K} + \frac{a_1K(2a_1+a_2)(M-1)TO}{(K+1)(2K+2Ma_1+Ma_2}. \quad (62)$$

5 Extensions

Lack of space has prevented us from discussing a number of interesting aspects of our work. One of these is an approximation method for determining the autocorrelation functions associated with the buffer lengths. Although the autocorrelation can not be computed in closed form, sample path descriptions facilitate the development of effective approximations based on matching key moments which can be computed explicitly. This type of information gives considerable insight into the effects of cascading systems

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