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Pattern generation, topology, and non-holonomic systems

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Received 5 October 2004; accepted 15 February 2005

Abstract

We consider the problem of achieving a desired steady-state effect through periodic behavior for a class of control systems with and without drift. The problem of using periodic behavior to achieve set-point regulation for the control systems with drift is directly related to that of achieving unbounded effect for the corresponding driftless control systems. We prove that in both cases, the ability to use periodic behavior, and more generally, bounded behavior, to achieve the desired goal implies, under a certain topological condition, the non-holonomicity of the control systems. We also prove that under a regularity condition, the resulting system trajectories must be area-generating in a precise sense.

Keywords: Non-holonomic systems; Simple connectedness; Pattern generation

1. Introduction

Consider the control system

$$\begin{aligned}\dot{x}_1 &= u_1,\\ \dot{x}_2 &= u_2,\\ \dot{x}_3 &= x_1u_2 - x_2u_1 - \alpha x_3,\end{aligned} \tag{1}$$

where $x_1, x_2, x_3 \in \mathbb{R}$ and $\alpha > 0$. Does this system admit a stable trajectory[1] with x_1, x_2 time-periodic and x_3 equal to an arbitrary constant $e > 0$? In other words, can x_3 be stabilized to $e > 0$ through periodic behavior in x_1, x_2? It is shown in [1] that the feedback control law defined by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\omega x_2 + \beta(e - x_3)x_1 \\ \omega x_1 + \beta(e - x_3)x_2 \end{pmatrix}$$

gives rise to a closed-loop system which admits a one-parameter family of orbitally asymptotically stable periodic solutions

$$x_p(t) = \begin{pmatrix} \sqrt{\dfrac{\alpha e}{\omega}}\cos(\omega t + \phi) \\ \sqrt{\dfrac{\alpha e}{\omega}}\sin(\omega t + \phi) \\ e \end{pmatrix}, \quad 0 \leqslant \phi \leqslant 2\pi,$$

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[1] Throughout this paper, by trajectory, we mean *continuous* trajectory.

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doi:10.1016/j.sysconle.2005.02.007

realizing set point control through oscillatory behavior. A generalization of (1) is given by

$$\dot{x} = u,$$
$$\dot{X} = xu^{\mathrm{T}} - ux^{\mathrm{T}} - \alpha X, \qquad (2)$$

where $x \in \mathbb{R}^k$ and $X \in so(k)$; here as well, a control law is given in [1] that yields a closed-loop system which admits stable quasi-periodic solutions.

Similar questions can be formulated for driftless control systems as well. Consider the driftless version of control system (1), defined by

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = u_2,$$
$$\dot{x}_3 = x_1 u_2 - x_2 u_1, \qquad (3)$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Does this system have trajectories with (x_1, x_2) time-periodic and x_3 unbounded? Clearly, one such trajectory is given through the feedback control law $u_1 = -x_2, u_2 = x_1$. Indeed, with this control law, x_1, x_2 are periodic functions of time, $x_1^2 + x_2^2$ is constant, and $x_3(t) = (x_1^2 + x_2^2)t + x_3(0)$. Hence, except for the trivial trajectory with initial conditions $x_1 = x_2 = 0$, x_3 is unbounded; in other words, an "unbounded result" (x_3) is obtained through a "periodic action" (x_1, x_2). Consider now instead the control system

$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = u_2,$$
$$\dot{x}_3 = x_1 u_1 + x_2 u_2, \qquad (4)$$

obtained by slightly modifying control system (3). It is easy to see that the trajectories of this system are such that x_3 and $x_1^2 + x_2^2$ differ only by a constant. Hence, there can be no trajectories with x_1 and x_2 time-periodic and x_3 unbounded.

It is worth to note at this point that systems (1), (2) and (3) are non-holonomic, whereas system (4) is holonomic. This and similar observations lead to the statement in [1] that "the appearance of time-periodic phenomena in both man-made and biological systems can often be traced to non-integrable effects of the type that arises in nonlinear controllability". The goal of this paper is to make this statement precise by proving that under a certain topological condition, non-holonomicity is necessary for the trajectories considered above to arise, and that the key property of these trajectories (which implies the non-holonomicity of the control system) is not the time-periodicity of a distinguished subset of their components, but rather their boundedness. We shall also prove that these same trajectories have to be area-generating, in a precise sense to be described.

2. Non-integrability and topology

Let $m, n \in \mathbb{N}^\star$ and let Ω be an m-dimensional connected manifold. For $i = 1, \ldots, m$, let

$$g_i : \Omega \times \mathbb{R}^n \to T(\Omega \times \mathbb{R}^n),$$
$$(x_a, x_b) \mapsto g_i(x_a, x_b)$$

be C^∞ vector fields on $\Omega \times \mathbb{R}^n$, and let $\mathscr{D}$ denote the distribution on $\Omega \times \mathbb{R}^n$ spanned by these vector fields, that is, for all $(x_a, x_b) \in \Omega \times \mathbb{R}^n$, $\mathscr{D}(x_a, x_b)$ is the vector subspace of $T_{(x_a,x_b)}(\Omega \times \mathbb{R}^n)$ spanned by $\{g_i(x_a, x_b)\}_{i=1}^m$. Let

$$\pi_a : \Omega \times \mathbb{R}^n \to \Omega,$$
$$(x_a, x_b) \mapsto x_a$$

be the projection onto the first component. In all that follows, we shall assume:

1. The differential $\mathrm{d}\pi_a|_{(x_a,x_b)} : \mathscr{D}(x_a, x_b) \to T_{x_a}\Omega$ is a vector space isomorphism for all $(x_a, x_b) \in \Omega \times \mathbb{R}^n$.
2. For all $i = 1, \ldots, m$, g_i has no dependence on x_b, that is, there exist C^∞ mappings $\tilde{g}_i : \Omega \to T(\Omega \times \mathbb{R}^n)$ such that $g_i = \tilde{g}_i \circ \pi_a$.

We shall, with some abuse of notation, henceforth write $g_i(x_a)$ instead of $g_i(x_a, x_b)$ and we shall exclusively consider the following classes of control systems.

Definition 1. We define a driftless affine control system by

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_b \end{pmatrix} = \sum_{i=1}^{m} g_i(x_a) u_i,$$

and an affine control system with partial drift by

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_b \end{pmatrix} = \begin{pmatrix} 0 \\ c(x_b) \end{pmatrix} + \sum_{i=1}^{m} g_i(x_a)u_i,$$

where $c : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and for $i=1, \ldots, m$, the control functions u_i are functions of x_a, x_b and t.

Note that the control system in Eq. (1) is a control system with partial drift with $m = 2$, $n = 1$, $\Omega = \mathbb{R}^2$, and with

$$g_1(x_1, x_2) = \begin{pmatrix} 1 \\ 0 \\ -x_2 \end{pmatrix}, \quad g_2(x_1, x_2) = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix},$$
$$c(x_3) = -\alpha x_3.$$

Similarly for the control system in Eq. (2), with $m=k$, $n = k(k-1)/2$, and $\Omega = \mathbb{R}^k$.

Consider the driftless control system in Definition 1; The following result specifies the linkage between generating unbounded result (x_b) with bounded action (x_a), the non-holonomicity of the control system, and the topology of the domain Ω of the x_a.

Theorem 1. *Consider the driftless control system given in Definition* 1, *and assume there exists a trajectory* $(x_a, x_b) : \mathbb{R}^+ \to \Omega \times \mathbb{R}^n$ *such that* $(x_b(t))_{t\in\mathbb{R}^+}$ *is an unbounded subset of* $\mathbb{R}^n$ *and* $x_a(t) \in K$, *for all* $t \in \mathbb{R}^+$, *where K is a compact subset of* Ω. *Then*

Ω *simply-connected* $\Longrightarrow \mathscr{D}$ *non-integrable.*

Remark. Theorem 1 highlights a topological constraint to generating unbounded result (the x_b variable) using bounded action (the x_a variable). It is an interesting exercise to ponder the significance of this theorem in light of the following simple example: The rolling directed vertical coin on a planar surface covers an unbounded distance through bounded action (the configuration space of the coin is S^1, hence compact), though the rolling constraint is holonomic. In this example, we have $\Omega = S^1$, $m = 1$, $n = 2$, and the vector field g spanning the distribution $\mathscr{D}$ is given by

$$g = r \cos\phi \frac{\partial}{\partial x} + r \sin\phi \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta},$$

where θ is the angular parameter of the coin, ϕ is the angle the vertical coin makes with the x-axis, r is the radius of the coin, and x, y are planar Cartesian coordinates. It is clear that $\mathscr{D}$ satisfies all the assumptions leading to Theorem 1. Since $m = 1$, the distribution $\mathscr{D}$ is integrable. Furthermore, S^1 is compact. It follows therefore from Theorem 1 that the configuration space S^1 of the coin must be multiply-connected, and indeed the fundamental group of S^1 is $\mathbb{Z}$.

Proof of Theorem 1. Assume Ω is simply-connected and $\mathscr{D}$ is integrable; we shall derive a contradiction. $\mathscr{D}$ being assumed integrable, let M be the maximal integral manifold of $\mathscr{D}$ containing $(x_a(0), x_b(0)) \in \Omega \times \mathbb{R}^n$. We have:

Proposition 1. *The mapping* $\pi_a : M \to \Omega$ *is a covering map.*

Proof. We first prove that $\pi_a : M \to \Omega$ is surjective. Let $q \in \pi_a(M)$, and let $p \in \pi_a^{-1}(q)$. Since $\mathrm{d}\pi_a : \mathscr{D}(x_a, x_b) \to T_{x_a}\Omega$ is a vector space isomorphism for all $(x_a, x_b) \in \Omega \times \mathbb{R}^n$, by the inverse function theorem there exists an open neighborhood $U_p \subset M$ for the integral manifold topology of M, and a neighborhood $U_q \subset \Omega$ such that the restriction $\pi_a|_{U_p} : U_p \to U_q$ is a diffeomorphism. This proves that the mapping $\pi_a : M \to \Omega$ is a local diffeomorphism. It then follows immediately that $\pi_a(M)$ is an open subset of Ω. Let now $q \in \overline{\pi_a(M)}$. Then there exists a sequence $(p_n)_n \subset M$ such that $\pi_a(p_n) \to q$ in Ω as $n \to \infty$. Let $z \in \mathbb{R}^n$; then $(q, z) \in \Omega \times \mathbb{R}^n$, and consider the maximal integral manifold $M_{(q,z)}$ of $\mathscr{D}$ containing (q, z). By the local diffeomorphism property of π_a, there exists an open subset $U_{(q,z)}$ of (q, z) in $M_{(q,z)}$ for the integral manifold topology of $M_{(q,z)}$, and an open subset U_q of q in Ω, such that the restriction $\pi_a|_{U_{(q,z)}} : U_{(q,z)} \to U_q$ is a diffeomorphism. Now, for all $y \in \mathbb{R}^n$, the image $U_{(q,z+y)}$ of $U_{(q,z)}$ under the diffeomorphism $(x_a, x_b) \mapsto (x_a, x_b + y)$ is an integral manifold of $\mathscr{D}$ containing the point $(q, z+y)$; furthermore, there exists $N \in \mathbb{N}$ such that $k \geqslant N \Rightarrow \pi_a(p_k) \in U_q$. In particular, $\pi_a(p_N) \in U_q$. Let $y \in \mathbb{R}^n$ such that $p_N \in U_{(q,z+y)}$. Such a y exists since $\pi_a|_{U_{(q,z)}} : U_{(q,z)} \to U_q$ is a diffeomorphism, and $\pi_a(p_N) \in U_q$. Then $\tilde{M} = M \cup U_{(q,z+y)}$ is a connected integral manifold of $\mathscr{D}$ containing $(x_a(0), x_b(0))$, and $M \subset \tilde{M}$. By maximality of M, we must have $\tilde{M} = M$, and hence $(q, z + y) \in M$, and therefore $q = \pi_a(q, z + y) \in \pi_a(M)$. Hence, $\pi_a(M)$ is a closed subset of Ω. Since Ω

is connected, $\pi_a(M)=\Omega$, and therefore $\pi_a|_M : M \to \Omega$ is a surjective mapping.

It follows from the above that every $q \in \Omega$ has an open neighborhood $U_q \subset \Omega$ such that

$$\pi_a^{-1}(U_q)=\bigcup_{\lambda\in\Lambda} U_\lambda,$$

where $\Lambda \subset \mathbb{R}^n$, $U_\lambda \subset M$ for all $\lambda \in \Lambda$, $U_{\lambda_1}\cap U_{\lambda_2}=\emptyset$ for $\lambda_1 \neq \lambda_2$, and U_λ is diffeomorphic to U_q for all $\lambda \in \Lambda$. The local diffeomorphism property and the surjectivity of π_a then imply that $\pi_a : M \to \Omega$ is a covering map ([3]). □

Consider now the identity map $\mathrm{id}_\Omega : \Omega \to \Omega$. Since Ω is, by assumption, simply-connected, we can lift id_Ω to a C^∞ map $f_\Omega : \Omega \to M$ such that $\pi_a \circ f_\Omega = \mathrm{id}_\Omega$ and $f_\Omega(x_a(0))=(x_a(0), x_b(0))$. Hence M is the graph of the C^∞ mapping $\pi_b \circ f_\Omega : \Omega \to \mathbb{R}^n$, where $\pi_b : \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the second component, and therefore $x_b=\pi_b\circ f_\Omega\circ x_a$. Since $x_a(t) \in K$, for all $t \in \mathbb{R}^+$, we have $x_b(t) \in \pi_b\circ f_\Omega(K)$, for all $t \in \mathbb{R}^+$. Since $\pi_b \circ f_\Omega$ is C^∞, and hence continuous, $\pi_b\circ f_\Omega(K)$ is a compact, hence bounded, subset of $\mathbb{R}^n$, and this contradicts the assumption that $(x_b(t))_{t\in\mathbb{R}^+}$ is an unbounded subset of $\mathbb{R}^n$. This concludes the proof of Theorem 1. □

In order to relate driftless systems to systems with partial drift, we prove the following lemma.

Lemma 1. *Let $c : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $y \in \mathbb{R}^n$ with $c(y) \neq 0$. Then, $\forall\ 0<\alpha<\frac{1}{2}$, there exists $t_0>0$ such that $\forall t \geqslant t_0$:*

$$\left\| \int_{t_0}^{t} c(x_b(\tau))\,\mathrm{d}\tau \right\| \geqslant \alpha(t-t_0)\|c(y)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Proof. Let $0<\alpha<\frac{1}{2}$; then $\sqrt{1-2\alpha}>0$, and since by assumption $x_b(t) \to y$ as $t \to \infty$, c is continuous, and $c(y) \neq 0$, there exists $t_0>0$ such that for all $t\geqslant t_0$ we have $\|c(x_b(t)) - c(y)\| \leqslant \sqrt{1-2\alpha}\|c(y)\|$. This then implies $\langle c(y), c(x_b(t)) - \alpha c(y)\rangle \geqslant 0$ for all $t\geqslant t_0$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$. Now,

$$\begin{aligned}
&\left\langle \int_{t_0}^{t} c(x_b(\tau))\,\mathrm{d}\tau, c(y)\right\rangle \\
&\quad = \int_{t_0}^{t} \langle c(x_b(\tau)) - \alpha c(y), c(y)\rangle\,\mathrm{d}\tau \\
&\qquad + \alpha(t-t_0)\|c(y)\|^2 \\
&\qquad \geqslant \alpha(t-t_0)\|c(y)\|^2,
\end{aligned}$$

and since $c(y) \neq 0$ by assumption, the desired inequality follows from Cauchy-Schwarz' inequality. □

The following result is the analog, for systems with partial drift, of Theorem 1.

Theorem 2. *Consider the control system with partial drift given in Definition* 1; *Assume there exists a trajectory $(x_a, x_b) : \mathbb{R}^+ \to \Omega\times\mathbb{R}^n$ such that $\{x_b=y\}$ is an asymptotically stable surface of that system, where $y \in \mathbb{R}^n$ and $c(y) \neq 0$, and $x_a(t) \in K$ for all $t \in \mathbb{R}^+$, where K is a compact subset of Ω; then*

Ω simply-connected $\implies \mathscr{D}$ non-integrable.

Proof. Assume $\pi_1(\Omega)=0$. Defining the function $y_b : \mathbb{R}^+ \to \mathbb{R}^n$ by $y_b(t)=x_b(t)-\int_0^t c(x_b(\tau))\,\mathrm{d}\tau$, we obtain the driftless control system:

$$\begin{aligned}
(\dot{x}_a(t), \dot{y}_b(t)) &= (\dot{x}_a(t), \dot{x}_b(t) - c(x_b(t))) \\
&= \sum_{i=1}^{m} g_i(x_a(t))u_i(x_a(t), x_b(t), t) \\
&= \sum_{i=1}^{m} g_i(x_a(t))u_i\left(x_a(t), y_b(t) \right.\\
&\qquad \left. + \int_0^t c(x_b(\tau))\,\mathrm{d}\tau, t\right) \\
&= \sum_{i=1}^{m} g_i(x_a(t))\tilde{u}_i(x_a(t), y_b(t), t),
\end{aligned}$$

where for $i=1,\dots,m$, the functions $\tilde{u}_i : \Omega\times\mathbb{R}^n\times\mathbb{R}^+ \to \mathbb{R}$ are the new controls. Now let $0<\alpha<\frac{1}{2}$.

By Lemma 1, there exists $t_0 > 0$ such that for all $t \geqslant t_0$:

$$\begin{aligned}\|y_b(t)\| \geqslant & \left\| \int_{t_0}^{t} c(x_b(\tau))\,\mathrm{d}\tau \right\| - \left\| \int_{0}^{t_0} c(x_b(\tau))\,\mathrm{d}\tau \right\| \\ & - \|x_b(t)\| \\ \geqslant & \alpha(t-t_0)\|c(y)\| - \left\| \int_{0}^{t_0} c(x_b(\tau))\,\mathrm{d}\tau \right\| \\ & - \|x_b(t)\|,\end{aligned}$$

and since $x_b(t) \to y$ and $c(y) \neq 0$ by assumption, we have $\limsup_{t\to\infty} \|y_b(t)\| = \infty$, which implies that $(y_b(t))_{t\in\mathbb{R}^+}$ is an unbounded subset of $\mathbb{R}^n$. It then follows from Theorem 1 that the distribution $\mathscr{D}$ is non-integrable. This concludes the proof of Theorem 2. □

3. Non-integrability and area generation

We shall now examine the particular trajectories associated with the control systems in Definition 1. The main property of such trajectories is that they have to be area-generating in a precise sense. We shall prove this property for driftless control systems; the proof carries over almost verbatim to control systems with partial drift, along the lines of the proof of Theorem 2. In what follows, we shall, with some abuse of notation, denote by the same symbol x_a both a point in Ω and its expression in some local coordinate system—we are, in effect, restricting ourselves to a coordinate chart on Ω. Let x_a^k (resp. x_b^k) denote the kth component of x_a (resp. x_b). We have:

Lemma 2. *The distribution $\mathscr{D}$ is defined by the vanishing of the exterior differential ideal $\mathscr{I}$ in $\Omega \times \mathbb{R}^n$ generated by the one forms $\{\omega^k\}_{k=1}^n$, with $\omega^k = \mathrm{d}x_b^k - \sum_{l=1}^m C_l^k \,\mathrm{d}x_a^l$, where the $C_l^k : \Omega \to \mathbb{R}$ are smooth functions.*

Proof. The vector fields $g_i : \Omega \to T(\Omega \times \mathbb{R}^n)$ spanning the distribution $\mathscr{D}$ can be written

$$g_i = \sum_{k=1}^m A_i^k \frac{\partial}{\partial x_a^k} + \sum_{k=1}^n B_i^k \frac{\partial}{\partial x_b^k},$$

where $A_i^k : \Omega \to \mathbb{R}$ $(k = 1, \ldots, m)$ and $B_i^k : \Omega \to \mathbb{R}$ $(k = 1, \ldots, n)$ are smooth functions. Let $\omega = \sum_{k=1}^m C_k \,\mathrm{d}x_a^k + \sum_{k=1}^n D_k \,\mathrm{d}x_b^k$ be a 1-form in $\Omega \times \mathbb{R}^n$. For ω to belong to the annihilator of $\mathscr{D}$ we need to have $\omega(g_i) = 0$ for all $i = 1, \ldots, m$, or, equivalently,

$$\sum_{k=1}^m A_i^k C_k + \sum_{k=1}^n B_i^k D_k = 0, \quad \forall i = 1, \ldots, m. \tag{5}$$

Let $\mathbf{A}$ be the $m \times m$ matrix with i, k entry A_i^k, and let $\mathbf{B}$ be the $m \times n$ matrix with i, k entry B_i^k; similarly, let C be the m-vector with kth entry C_k, and D the n-vector with kth entry D_k. The assumption on the distribution $\mathscr{D}$ that the differential of the projection map π_a restricted to $\mathscr{D}$ be a vector space isomorphism at each point implies that matrix $\mathbf{A}$ is invertible. Eq. (5) can therefore be rewritten as $C = \mathbf{A}^{-1}\mathbf{B}D$. Note that $\mathbf{A}$ and $\mathbf{B}$ are smooth matrix-valued functions defined on Ω. Now, for $l = 1, \ldots, n$, let D^l be the n-vector with lth entry -1 and all other entries 0, and let C^l be the m-vector $C^l = \mathbf{A}^{-1}\mathbf{B}D^l$. The one-form ω^l defined by $\omega^l = \mathrm{d}x_b^l - \sum_{k=1}^m C_k^l \,\mathrm{d}x_a^k$ is then in the annihilator of $\mathscr{D}$, with the C_k^l smooth real-valued functions on Ω. Furthermore, since the forms ω^l, $l = 1, \ldots, n$ are linearly independent at each point, and since $\mathscr{D}$ is a rank m distribution, $\mathscr{D}$ is defined by the vanishing of the exterior differential ideal $\mathscr{I}$ in $\Omega \times \mathbb{R}^n$ generated by the one forms $\{\omega^l\}_{l=1}^n$. □

Definition 2. The distribution $\mathscr{D}$ defined by the vanishing of the exterior differential ideal $\mathscr{I}$ generated by the 1-forms $\{\omega^k = \mathrm{d}x_b^k - \sum_{l=1}^m C_l^k \,\mathrm{d}x_a^l\}_{k=1}^n$ is called regular if for all $k \in \{1, \ldots, n\}$, the one-form $\sum_{l=1}^m C_l^k \,\mathrm{d}x_a^l$ has constant rank r_k in Ω.

Lemma 3. *Assume the distribution $\mathscr{D}$ is regular, and let $l \in \{1, \ldots, n\}$. Then there exists an open subset $V \subset \Omega$, an open neighborhood $U \subset \mathbb{R}^m$ of the origin (with coordinate functions $y_a^1, \ldots, y_a^m$), and a diffeomorphism $\Phi : U \times \mathbb{R}^n \to V \times \mathbb{R}^n$ such that:*

$$\Phi^\star(\omega^l) = \begin{cases} \mathrm{d}x_b^l - (y_a^1 \,\mathrm{d}y_a^2 + y_a^3 \,\mathrm{d}y_a^4 \\ \quad + \cdots y_a^{2s-1} \,\mathrm{d}y_a^{2s}), \quad r_l = 2s, \\ \mathrm{d}x_b^l - (y_a^1 \,\mathrm{d}y_a^2 + y_a^3 \,\mathrm{d}y_a^4 \\ \quad + \cdots \mathrm{d}y_a^{2s-1}), \quad r_l = 2s-1. \end{cases}$$

Proof. Since $\mathscr{D}$ is assumed regular, it follows from Darboux's normal form theorem [2,4], that there exist open neighborhoods $V \subset \Omega$, $U \subset \mathbb{R}^m$ (with coordinate functions $y_a^1, \ldots, y_a^m$), and a diffeomorphism $\phi : U \to V$ such that:

$$\phi^\star\left(\sum_{k=1}^{n} C_k^l(x_a)\,\mathrm{d}x_a^k\right) = \begin{cases} y_a^1\,\mathrm{d}y_a^2 + y_a^3\,\mathrm{d}y_a^4 + \cdots y_a^{2s-1}\,\mathrm{d}y_a^{2s}, & r_l = 2s,\\ y_a^1\,\mathrm{d}y_a^2 + y_a^3\,\mathrm{d}y_a^4 + \cdots \mathrm{d}y_a^{2s-1}, & r_l = 2s-1.\end{cases}$$

Define the diffeomorphism Φ by

$$\Phi : U \times \mathbb{R}^n \to V \times \mathbb{R}^n, \quad (y_a, x_b) \mapsto \Phi(y_a, x_b) = (\phi(y_a), x_b).$$

Then,

$$\begin{aligned}\Phi^\star(\omega^l) &= \Phi^\star\left(\mathrm{d}x_b^l - \sum_{k=1}^{m} C_k^l\,\mathrm{d}x_a^k\right)\\ &= \mathrm{d}(x_b^l \circ \Phi) - \Phi^\star\left(\sum_{k=1}^{m} C_k^l\,\mathrm{d}x_a^k\right)\\ &= \mathrm{d}x_b^l - \phi^\star\left(\sum_{k=1}^{m} C_k^l\,\mathrm{d}x_a^k\right)\\ &= \begin{cases} \mathrm{d}x_b^l - (y_a^1\,\mathrm{d}y_a^2 + y_a^3\,\mathrm{d}y_a^4 + \cdots y_a^{2s-1}\,\mathrm{d}y_a^{2s}), & r_l = 2s,\\ \mathrm{d}x_b^l - (y_a^1\,\mathrm{d}y_a^2 + y_a^3\,\mathrm{d}y_a^4 + \cdots \mathrm{d}y_a^{2s-1}), & r_l = 2s-1.\end{cases}\end{aligned}$$ □

We shall call the (not necessarily unique) open subset V of Ω provided by Lemma 3 "distinguished chart of Ω". We can now state the main result relating the nature of the trajectories arising from the control of non-holonomic systems—in essence, for non-holonomicity to be exploited, the trajectories have to be area-generating in a very precise sense.

Theorem 3. *Consider the driftless control system given in Definition* 1. *Assume Ω simply-connected and $\mathscr{D}$ regular, and assume there exists a piecewise-C^1 trajectory $(x_a, x_b) : t \mapsto (x_a(t), x_b(t))$ such that $x_a(t) \in K$, for all $t \in \mathbb{R}^+$, where K is a compact subset of a distinguished chart V of Ω, and $(x_b(t))_{t\in\mathbb{R}^+}$ is an unbounded subset of $\mathbb{R}^n$; then there exists an open neighborhood $U \subset \mathbb{R}^m$ of the origin (with coordinate functions $y_a^1, \ldots, y_a^m$), a diffeomorphism*

$$\psi : U \to V, \quad y_a \mapsto \psi(y_a)$$

and integers $i, j \in \{1, \ldots, m\}$, $i \neq j$, such that the planar curve obtained by projecting the curve $\psi^{-1} \circ \pi_a \circ \gamma$ onto the (y^i, y^j)-plane has infinite area.

Proof. Note first that by Theorem 1 the distribution $\mathscr{D}$ is non-integrable. Let $\gamma : t \mapsto (x_a(t), x_b(t))$ denote the piecewise-C^1 trajectory of the driftless control system; by $\gamma_{[t_0,t_1]}$ we shall denote the restriction of γ to $[t_0, t_1] \subset \mathbb{R}^+$. Since $\mathscr{D}$ is assumed to be a regular distribution, it is given by the vanishing of the one-forms $\{\omega^k = \mathrm{d}x_b^k - \sum_{l=1}^m C_l^k\,\mathrm{d}x_a^l\}_{k=1}^n$, where the one-forms $\sum_{l=1}^m C_l^k\,\mathrm{d}x_a^l$ have constant rank on Ω for all k. Hence, for all $l = 1, \ldots, n$:

$$x_b^l(t_1) - x_b^l(t_0) = \int_{\gamma_{[t_0,t_1]}} \sum_{k=1}^{n} C_k^l(x_a)\,\mathrm{d}x_a^k.$$

Since $(\|x_b(t)\|)_{t\in\mathbb{R}^+}$ is an unbounded subset of $\mathbb{R}$, there exists $l \in \{1, \ldots, n\}$ such that $(|x_b^l(t)|)_{t\in\mathbb{R}^+}$ is unbounded. Let Φ be the diffeomorphism provided by Lemma 3, and assume without loss of generality that the rank r_l is odd. Then

$$\Phi^\star(\omega^l) = \mathrm{d}x_b^l - \sum_{k=1}^{s-1} y_a^{2k-1}\,\mathrm{d}y_a^{2k} - \mathrm{d}y_a^{2s-1},$$

and therefore

$$x_b^l(t_1) - x_b^l(t_0) = y_a^{2s-1}(t_1) - y_a^{2s-1}(t_0) + \sum_{k=1}^{s-1} \int_{\psi^{-1}\circ\pi_a\circ\gamma_{[t_0,t_1]}} y_a^{2k-1}\,\mathrm{d}y_a^{2k}.$$

By assumption, $y_a(t) \in \psi^{-1}(K)$ for all $t \in \mathbb{R}^+$, and $\psi^{-1}(K)$ is a compact, hence bounded, subset of $\mathbb{R}^m$. Hence, since $(x_b^l(t))_{t\in\mathbb{R}^+}$ is unbounded, there exists an integer k such that $(\int_{\psi^{-1}\circ\pi_a\circ\gamma_{[t_0,t]}} y_a^{2k-1}\,\mathrm{d}y_a^{2k})_{t\in\mathbb{R}^+}$ is unbounded, which, by Stokes' theorem, is equivalent to the area of the projection of the trajectory $\psi^{-1} \circ \pi_a \circ \gamma$ on the (y_a^{2k-1}, y_a^{2k}) plane being unbounded. This concludes the proof of Theorem 3. □

4. Conclusion

We have proven that for a certain class of control systems, the appearance of time-periodic, and even more generally, bounded phenomena directly implies the non-holonomicity of the control systems, modulo a precise topological condition involving the fundamental group. We have also shown that subject to a regularity condition, the resulting system trajectories must be area-generating. Two main questions need to still be resolved:

1. Simple connectedness of Ω is a sufficient condition which allowed lifting to the covering space; however, it is not always a necessary condition. Is it possible to refine the topological condition on Ω? If so, this would allow the extension of Theorem 1 to the case of a rolling rigid body, where $\Omega = SO(3)$.
2. How does the regularity condition on the distribution translate into properties of the vector fields defining the distribution?

Acknowledgements

The author is indebted to Professor Roger Brockett for numerous valuable suggestions that have improved this paper and for pointing out the relation between Theorem 1 and Poincaré's lemma.

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